

Large affine spaces of non-singular matrices

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Abstract

Let \mathbb{K} be an arbitrary (commutative) field with at least three elements. It was recently proven that an affine subspace of $M_n(\mathbb{K})$ consisting only of non-singular matrices must have a dimension lesser than or equal to $\binom{n}{2}$. Here, we classify, up to equivalence, the subspaces whose dimension equals $\binom{n}{2}$. This is done by classifying, up to similarity, all the $\binom{n}{2}$ -dimensional linear subspaces of $M_n(\mathbb{K})$ consisting of matrices with no non-zero invariant vector, reinforcing a classical theorem of Gerstenhaber. Both classifications only involve the quadratic structure of the field \mathbb{K} .

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1 Introduction

1.1 Introduction and basic definitions

In this article, we let \mathbb{K} be an arbitrary (commutative) field. We denote by $M_n(\mathbb{K})$ the algebra of square matrices with n rows and entries in \mathbb{K} , and by $GL_n(\mathbb{K})$ its group of invertible elements. We also denote by $M_{n,p}(\mathbb{K})$ the vector space of matrices with n rows, p columns and entries in \mathbb{K} . The transpose of a matrix M is denoted by M^T .

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An **affine** subspace \mathcal{V} of $M_n(\mathbb{K})$ is the translate of a linear subspace V of $M_n(\mathbb{K})$: then V is uniquely determined by \mathcal{V} (it is the set of all matrices M such that $M + \mathcal{V} = \mathcal{V}$) and is called the **translation vector space** of \mathcal{V} .

Given two linear (or affine) subspaces V and W of $M_n(\mathbb{K})$, we say that V and W are equivalent, and we write $V \sim W$, if $W = PVQ$ for some $(P, Q) \in GL_n(\mathbb{K})^2$; we say that V and W are similar, and we write $V \simeq W$, if $W = PVP^{-1}$ for some $P \in GL_n(\mathbb{K})$.

Two matrices A and B of $M_n(\mathbb{K})$ are called **congruent**, and we write $A \approx B$, if $A = PBP^T$ for some $P \in GL_n(\mathbb{K})$. Finally, two quadratic forms q and q' on vector spaces over \mathbb{K} are called **similar** if q' is equivalent to λq for some $\lambda \in \mathbb{K} \setminus \{0\}$.

Here, we are concerned with the geometry of $GL_n(\mathbb{K}) \cup \{0\}$ as a cone in the vector space $M_n(\mathbb{K})$. From the linear algebraist's viewpoint, the natural questions that one may ask are the following ones:

- What is the minimal linear (resp. affine) subspace of $M_n(\mathbb{K})$ containing $GL_n(\mathbb{K})$?
- What is the minimal linear subspace of $M_n(\mathbb{K})$ containing $M_n(\mathbb{K}) \setminus GL_n(\mathbb{K})$?
- What are the maximal linear (resp. affine) subspaces included in $GL_n(\mathbb{K}) \cup \{0\}$?
- What are the maximal linear (resp. affine) subspaces included in $M_n(\mathbb{K}) \setminus GL_n(\mathbb{K})$?

The first two problems have easy answers: $GL_n(\mathbb{K})$ always spans $M_n(\mathbb{K})$, the affine subspace it generates is $M_n(\mathbb{K})$ unless $n = 1$ and $\#\mathbb{K} = 2$, and $M_n(\mathbb{K}) \setminus GL_n(\mathbb{K})$ spans $M_n(\mathbb{K})$ unless $n = 1$.

The last two questions have no clear answer however and depend widely on the field \mathbb{K} . For example, $GL_n(\mathbb{C}) \cup \{0\}$ contains no 2-dimensional linear subspace, whilst $GL_{2n}(\mathbb{R}) \cup \{0\}$ always does. As for singular linear subspaces (i.e. linear subspaces included in $M_n(\mathbb{K}) \setminus GL_n(\mathbb{K})$), a classification of them is generally considered to be out of reach, even for an algebraically closed field, although a lot of progress has been made in understanding their structure in the 1980's (see the works of Atkinson, Lloyd and Stephens [1, 2, 3, 4] and our recent [12]).

Rather than try to classify all the linear (resp. affine) subspaces contained in $GL_n(\mathbb{K})$ or $M_n(\mathbb{K}) \setminus GL_n(\mathbb{K})$, a more modest approach is to find the maximal dimension for such a subspace and to classify the linear (resp. affine) subspaces

with a maximal *dimension*. To this day, this problem has been almost entirely solved:

- A linear subspace included in $\mathrm{GL}_n(\mathbb{K}) \cup \{0\}$ has dimension at most n ; linear subspaces in $\mathrm{GL}_n(\mathbb{K}) \cup \{0\}$ with dimension n correspond to the structures of (possibly non-associative and non-unital) division algebras on \mathbb{K}^n that are compatible with its vector space structure (see e.g. the last section of [13]). Note that no such subspace exists when $n \geq 2$ and \mathbb{K} is algebraically closed.
- An affine subspace included in $\mathrm{M}_n(\mathbb{K}) \setminus \mathrm{GL}_n(\mathbb{K})$ has dimension at most $n(n-1)$. If its dimension is $n(n-1)$, then it is equivalent to the space of matrices with zero as last column or to its transpose (unless $n = 2$ and $\#\mathbb{K} = 2$ in which case there is an additional equivalence class). This is a classical result of Dieudonné [5] (see also [11] for a simplified proof) which may be used to classify the endomorphisms of the vector space $\mathrm{M}_n(\mathbb{K})$ that stabilize $\mathrm{GL}_n(\mathbb{K})$ (see [13]).

Here, we will focus on the affine subspaces of $\mathrm{M}_n(\mathbb{K})$ that are included in $\mathrm{GL}_n(\mathbb{K})$. Let \mathcal{V} be such a subspace, and choose $P \in \mathcal{V}$. Then $P^{-1}\mathcal{V}$ is also included in $\mathrm{GL}_n(\mathbb{K})$, contains the identity matrix I_n and has the same dimension as \mathcal{V} . Denoting by H its translation vector space, we see that $I_n - \lambda M \in \mathrm{GL}_n(\mathbb{K})$ for every $\lambda \in \mathbb{K}$ and $M \in H$, hence the linear subspace H has the two following equivalent properties:

- For every $M \in H$, one has $\mathrm{Sp}(M) \subset \{0\}$, where $\mathrm{Sp}(M)$ denotes the set of eigenvalues of M in the field \mathbb{K} .
- No matrix of H possesses a non-zero invariant vector in \mathbb{K}^n .

Definition 1. A linear subspace H of $\mathrm{M}_n(\mathbb{K})$ is said to have a **trivial spectrum** if no matrix of H possesses a non-zero invariant vector in \mathbb{K}^n .

Note that for such a linear subspace H with a trivial spectrum, the affine subspace $I_n + H$ is included in $\mathrm{GL}_n(\mathbb{K})$, and so is any subspace equivalent to it. For example, if we denote by $\mathrm{NT}_n(\mathbb{K})$ the space of strictly upper triangular matrices of $\mathrm{M}_n(\mathbb{K})$, then $I_n + \mathrm{NT}_n(\mathbb{K})$ is an affine subspace of non-singular matrices with dimension $\binom{n}{2}$.

It follows that classifying up to equivalence the affine subspaces of non-singular matrices essentially amounts to classifying up to similarity the linear

subspaces of $M_n(\mathbb{K})$ with a trivial spectrum. In the case \mathbb{K} is algebraically closed, the linear subspaces with a trivial spectrum are the linear subspaces of nilpotent matrices: a famous theorem of Gerstenhaber [6] states that the dimension of such a subspace is bounded above by $\binom{n}{2}$ and that equality occurs only for subspaces similar to $NT_n(\mathbb{K})$. It is only very recently that the upper bound $\binom{n}{2}$ has been shown to apply to linear subspaces with a trivial spectrum for an arbitrary field (see the works of Quinlan [8] and our own [10]):

Theorem 1. *Let V be a linear subspace of $M_n(\mathbb{K})$ with a trivial spectrum. Then $\dim V \leq \binom{n}{2}$.*

Definition 2. A linear subspace of $M_n(\mathbb{K})$ with a trivial spectrum is called **maximal**¹ if its dimension is $\binom{n}{2}$.

Our aim is to classify the maximal linear subspaces of $M_n(\mathbb{K})$ with a trivial spectrum. Unlike the case of nilpotent linear subspaces, the structure of the ground field \mathbb{K} plays a large part in this classification. For example, if there exists a polynomial $t^2 - at - b \in \mathbb{K}[t]$ with degree two and no root in \mathbb{K} , then the line spanned by the companion matrix $\begin{bmatrix} 0 & b \\ 1 & a \end{bmatrix}$ is obviously a maximal linear subspace of $M_2(\mathbb{K})$ with a trivial spectrum and it is not similar to $NT_2(\mathbb{K})$. Another example is given by the space $A_n(\mathbb{R})$ of skew-symmetric real matrices, which has a trivial spectrum and dimension $\binom{n}{2}$, although it is not similar to $NT_n(\mathbb{R})$ if $n \geq 2$.

1.2 Reducibility

Notation 3. Let V and W be respective subsets of $M_n(\mathbb{K})$ and $M_p(\mathbb{K})$. Set

$$V \vee W := \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \mid (A, B, C) \in V \times M_{n,p}(\mathbb{K}) \times W \right\} \subset M_{n+p}(\mathbb{K}).$$

Note that if V and W are maximal linear subspaces with a trivial spectrum, then $V \vee W$ is a linear subspace with a trivial spectrum and dimension $\binom{n}{2} + \binom{p}{2} + pn = \binom{n+p}{2}$, hence it is maximal. Notice also that the composition law \vee is associative.

¹This should not be confused with the concept of maximality in the set of linear subspaces with a trivial spectrum ordered by the inclusion of subsets.

Definition 4. A maximal linear subspace of $M_n(\mathbb{K})$ with a trivial spectrum is called **irreducible** if the only linear subspaces of \mathbb{K}^n it stabilizes are $\{0\}$ and \mathbb{K}^n (and we call it reducible otherwise).

Conversely, let H be a maximal linear subspace of $M_n(\mathbb{K})$ with a trivial spectrum. Assume that there is a $p \in \llbracket 1, n-1 \rrbracket$ such that $F := \mathbb{K}^p \times \{0\}$ is stabilized by every matrix of H . Then we may write every matrix of H as

$$M = \begin{bmatrix} f(M) & g(M) \\ 0 & h(M) \end{bmatrix} \quad \text{for some } (f(M), g(M), h(M)) \in M_p(\mathbb{K}) \times M_{p, n-p}(\mathbb{K}) \times M_{n-p}(\mathbb{K}).$$

Therefore $V := f(H)$ and $W := h(H)$ are linear subspaces respectively of $M_p(\mathbb{K})$ and $M_{n-p}(\mathbb{K})$, each with a trivial spectrum, and since

$$\binom{n}{2} = \dim H \leq \dim V + \dim W + \dim g(H) \leq \binom{p}{2} + \binom{n-p}{2} + p(n-p) = \binom{n}{2},$$

we find that both V and W are maximal. Hence $H \subset V \vee W$, and since the dimensions are equal, we deduce that $H = V \vee W$.

Conjugating H with an appropriate invertible matrix, this generalizes as follows: if H is not irreducible, then $H \simeq V \vee W$ for some maximal linear subspaces V and W with trivial spectra. This yields:

Proposition 2. *Let H be a maximal linear subspace of $M_n(\mathbb{K})$ with a trivial spectrum. Then there are irreducible maximal linear subspaces V_1, \dots, V_p with trivial spectra such that*

$$H \simeq V_1 \vee V_2 \vee \dots \vee V_p.$$

This suggests that we focus our attention on the irreducible maximal subspaces.

1.3 Main theorems

Denote by $A_n(\mathbb{K})$ the set of *alternate* matrices of $M_n(\mathbb{K})$, i.e. the skew-symmetric ones with a zero diagonal, i.e. the ones for which $\forall X \in \mathbb{K}^n, X^T A X = 0$.

Definition 5. A matrix $P \in M_n(\mathbb{K})$ is called **non-isotropic** if the quadratic form $X \mapsto X^T P X$ is non-isotropic, i.e. $\forall X \in \mathbb{K}^n \setminus \{0\}, X^T P X \neq 0$.

Notice, in that case, that P is non-singular and that P^{-1} is non-isotropic. The subspace $P A_n(\mathbb{K})$ then has dimension $\binom{n}{2}$ and has a trivial spectrum: indeed, given $A \in A_n(\mathbb{K})$ and $X \in \mathbb{K}^n$,

$$P A X = X \Rightarrow P^{-1} X = A X \Rightarrow X^T P^{-1} X = 0 \Rightarrow X = 0.$$

We may now state our main results.

Theorem 3. Assume that $\#\mathbb{K} \geq 3$. Let n be a positive integer. Then the irreducible maximal linear subspaces of $M_n(\mathbb{K})$ with a trivial spectrum are the subspaces of the form $P A_n(\mathbb{K})$ for a non-isotropic matrix $P \in GL_n(\mathbb{K})$.

Theorem 4 (Classification theorem for maximal linear subspaces with a trivial spectrum). Assume that $\#\mathbb{K} \geq 3$. Let V be a maximal linear subspace of $M_n(\mathbb{K})$ with a trivial spectrum. Then there is a list $(P_1, \dots, P_p) \in GL_{n_1}(\mathbb{K}) \times \dots \times GL_{n_p}(\mathbb{K})$ of non-isotropic matrices such that

$$V \simeq P_1 A_{n_1}(\mathbb{K}) \vee \dots \vee P_p A_{n_p}(\mathbb{K}).$$

The integer p is uniquely determined by V and, for every $k \in \llbracket 1, p \rrbracket$, the matrix P_k is uniquely determined by V up to congruence and multiplication by a non-zero scalar. Moreover, given another list $(Q_1, \dots, Q_p) \in GL_{n_1}(\mathbb{K}) \times \dots \times GL_{n_p}(\mathbb{K})$, if Q_k is congruent to a scalar multiple of P_k for each $k \in \llbracket 1, p \rrbracket$, then

$$V \simeq Q_1 A_{n_1}(\mathbb{K}) \vee \dots \vee Q_p A_{n_p}(\mathbb{K}).$$

If \mathbb{K} is quadratically closed, it follows that there is no irreducible maximal linear subspace of $M_n(\mathbb{K})$ with a trivial spectrum for $n \geq 2$. If \mathbb{K} is finite (with at least three elements), then every 3-dimensional quadratic form over \mathbb{K} is isotropic, hence $M_n(\mathbb{K})$ contains no irreducible maximal linear subspace with a trivial spectrum for $n \geq 3$. We deduce the following corollaries:

Corollary 5. Let \mathbb{K} be a quadratically closed field. Then $NT_n(\mathbb{K})$ is, up to similarity, the sole maximal linear subspace of $M_n(\mathbb{K})$ with a trivial spectrum.

Corollary 6. Let \mathbb{K} be a finite field with at least three elements. Let V be a maximal linear subspace of $M_n(\mathbb{K})$ with a trivial spectrum.

Then there are matrices M_1, \dots, M_p , either equal to $0 \in M_1(\mathbb{K})$ or belonging to $M_2(\mathbb{K})$ with no eigenvalue in \mathbb{K} , such that

$$V \simeq \mathbb{K} M_1 \vee \dots \vee \mathbb{K} M_p.$$

Each M_k is then uniquely determined by V up to similarity and multiplication by a non-zero scalar.

We may finally state the structure theorem for affine subspaces of non-singular matrices.

Theorem 7 (Classification theorem for large affine subspaces of non-singular matrices). *Assume that $\#\mathbb{K} \geq 3$. Let \mathcal{V} be a $\binom{n}{2}$ -dimensional affine subspace of $M_n(\mathbb{K})$ included in $GL_n(\mathbb{K})$. Then there is a list $(P_1, \dots, P_p) \in GL_{n_1}(\mathbb{K}) \times \dots \times GL_{n_p}(\mathbb{K})$ of non-isotropic matrices such that $n = n_1 + \dots + n_p$ and*

$$\mathcal{V} \sim I_n + (P_1 A_{n_1}(\mathbb{K}) \vee \dots \vee P_p A_{n_p}(\mathbb{K})).$$

The integer p is uniquely determined by \mathcal{V} and, for $1 \leq k \leq p$, the similarity class of the non-isotropic quadratic form $X \mapsto X^T P_k X$ is uniquely determined by \mathcal{V} . Moreover, given another list $(Q_1, \dots, Q_p) \in GL_{n_1}(\mathbb{K}) \times \dots \times GL_{n_p}(\mathbb{K})$, if $X \mapsto X^T Q_k X$ is similar to $X \mapsto X^T P_k X$ for each $k \in \llbracket 1, p \rrbracket$, then

$$\mathcal{V} \sim I_n + (Q_1 A_{n_1}(\mathbb{K}) \vee \dots \vee Q_p A_{n_p}(\mathbb{K})).$$

Note that the existence of (P_1, \dots, P_p) is a trivial consequence of Theorem 4 using the considerations of Paragraph 1.1.

As a consequence, $\binom{n}{2}$ -dimensional affine subspaces of $M_n(\mathbb{K})$ included in $GL_n(\mathbb{K})$ are classified, up to equivalence, by the lists of the form $([\varphi_1], \dots, [\varphi_p])$ where the φ_k 's are finite-dimensional non-isotropic quadratic forms over \mathbb{K} , the $[\varphi_k]$'s are their similarity classes, and $\sum_{k=1}^p \dim \varphi_k = n$. For the field of real numbers, this has the following striking corollary:

Corollary 8. *Let \mathcal{V} be an affine subspace of $M_n(\mathbb{R})$ included in $GL_n(\mathbb{R})$ with dimension $\binom{n}{2}$. Then there is a unique list (n_1, \dots, n_p) of positive integers such that $n = n_1 + \dots + n_p$ and*

$$\mathcal{V} \sim I_n + (A_{n_1}(\mathbb{R}) \vee \dots \vee A_{n_p}(\mathbb{R})).$$

1.4 Totally intransitive action of a space of matrices

Proving the previous theorems will require an extensive use of the following concept and of the subsequent remark:

Definition 6. Let V be a linear subspace of $M_n(\mathbb{K})$. For $X \in \mathbb{K}^n$, set

$$VX := \{MX \mid X \in V\}.$$

Note that VX is always a linear subspace of \mathbb{K}^n .

We say that V acts **totally intransitively** on \mathbb{K}^n if $VX \neq \mathbb{K}^n$ for every $X \in \mathbb{K}^n$, which is equivalent to having $\dim(VX) < n$ for every $X \in \mathbb{K}^n$.

Remark 1. If V has a trivial spectrum, then $X \notin VX$ for every $X \in \mathbb{K}^n \setminus \{0\}$, hence V acts totally intransitively on \mathbb{K}^n .

Moreover $V^T := \{M^T \mid M \in V\}$ also has a trivial spectrum, hence

$$\forall X \in \mathbb{K}^n, \quad \dim(VX) < n \quad \text{and} \quad \dim(V^T X) < n.$$

1.5 Structure of the paper

We will start (Section 2) with general considerations on the spaces of the type $PA_n(\mathbb{K})$ with $P \in \text{GL}_n(\mathbb{K})$. Using some of the obtained results, we will then prove the uniqueness statements in Theorems 4 and 7 (Section 3). The proof of Theorem 3 will be carried out in Section 4 by induction on n , starting from $n = 2$ and using a recent lemma that was proved in [10]: this is, by far, the most technical part of the paper. In Section 5, we will easily derive Gerstenhaber's theorem from Theorem 4 in the case $\#\mathbb{K} \geq 3$. In Section 6, we will show that Theorem 3 fails for $n = 3$ and $\mathbb{K} \simeq \mathbb{F}_2$. The case $\#\mathbb{K} = 2$ remains a very exciting challenge that we will not undertake here.

2 Basic properties of the spaces $PA_n(\mathbb{K})$

We consider first $PA_n(\mathbb{K})$ for an arbitrary $P \in \text{GL}_n(\mathbb{K})$. To start with, note that, for every $Q \in \text{GL}_n(\mathbb{K})$, one has

$$PA_n(\mathbb{K})Q = P(Q^T)^{-1}Q^T A_n(\mathbb{K})Q = (P(Q^T)^{-1}) A_n(\mathbb{K})$$

which immediately shows that $\{PA_n(\mathbb{K}) \mid P \in \text{GL}_n(\mathbb{K})\}$ is an equivalence class (for the equivalence of spaces of matrices).

In order to move forward, we need some basic properties of $A_n(\mathbb{K})$: for this, we equip \mathbb{K}^n with the non-degenerate symmetric bilinear form $(X, Y) \mapsto X^T Y$.

Lemma 9. *For any $X \in \mathbb{K}^n \setminus \{0\}$, one has*

$$A_n(\mathbb{K})X = \{X\}^\perp$$

and in particular $\dim(A_n(\mathbb{K})X) = n - 1$.

Proof. This is obvious if X is the first vector e_1 of the canonical basis of \mathbb{K}^n . In the general case, we may find some $P \in \text{GL}_n(\mathbb{K})$ such that $Pe_1 = X$, and note

that

$$\begin{aligned} A_n(\mathbb{K})X &= (P^T)^{-1}P^T A_n(\mathbb{K})Pe_1 = (P^T)^{-1}A_n(\mathbb{K})e_1 \\ &= (P^T)^{-1}\{e_1\}^\perp = \{Pe_1\}^\perp = \{X\}^\perp. \end{aligned}$$

□

We may now determine, amongst the spaces of the above form, those with a trivial spectrum:

Lemma 10. *Let $P \in \text{GL}_n(\mathbb{K})$. Then $PA_n(\mathbb{K})$ has a trivial spectrum if and only if P is non-isotropic.*

Proof. The “if” part has already been dealt with in the beginning of Section 1.3. Assume that P is isotropic. Then obviously $(P^T)^{-1}$ is also isotropic, hence we find a non-zero vector $X \in \mathbb{K}^n$ such that $X^T(P^T)^{-1}X = 0$, i.e. $P^{-1}X \in \{X\}^\perp$. Then Lemma 9 shows that $P^{-1}X = AX$ for some $A \in A_n(\mathbb{K})$ hence $(PA)X = X$, which shows that $PA_n(\mathbb{K})$ does not have a trivial spectrum. □

Proposition 11. *Let $P \in \text{GL}_n(\mathbb{K})$ be a non-isotropic matrix. Then $PA_n(\mathbb{K})$ is an irreducible maximal subspace with a trivial spectrum.*

Proof. It only remains to show that $PA_n(\mathbb{K})$ is irreducible. We use a *reductio ad absurdum* by assuming that it has a non-trivial stable subspace $F \subset \mathbb{K}^n$ with dimension $p \in \llbracket 1, n-1 \rrbracket$. Then F^\perp is stabilized by $(PA_n(\mathbb{K}))^T = A_n(\mathbb{K})P^T$. Choosing an arbitrary non-zero vector $X \in F$, we have $\dim(PA_n(\mathbb{K})X) = \dim\{X\}^\perp = n-1$ hence $p = n-1$.

However, choosing a non-zero vector $Y \in F^\perp$ yields $\dim(A_n(\mathbb{K})P^TY) = n-1$ hence $n-p = n-1$. This yields $n = 2$ and $p = 1$, in which case every matrix of $PA_n(\mathbb{K})$ must be nilpotent (since it has an eigenvector and 0 is the sole possible eigenvalue in \mathbb{K}), contradicting the fact that every non-zero matrix of $PA_2(\mathbb{K})$ is non-singular. □

We now investigate when two spaces of the form $PA_n(\mathbb{K})$ are similar. Here is our basic result:

Lemma 12. *Let $P \in \text{GL}_n(\mathbb{K})$. Then $PA_n(\mathbb{K}) = A_n(\mathbb{K})$ if and only if P is a scalar multiple of the identity.*

Proof. The “if” part is trivial. Assume conversely that $PA_n(\mathbb{K}) = A_n(\mathbb{K})$. Let $X \in \mathbb{K}^n \setminus \{0\}$. Then $PA_n(\mathbb{K})X = A_n(\mathbb{K})X$ yields that P stabilizes the hyperplane $\{X\}^\perp$, hence P^T stabilizes $\text{span}(X)$. Since this holds for every non-zero $X \in \mathbb{K}^n$, this shows that P^T is a scalar multiple of the identity, hence P also is. \square

The following corollary will be our starting point for the uniqueness statement in Theorem 4:

Proposition 13. *Let $(P, Q) \in \text{GL}_n(\mathbb{K})^2$. Then $PA_n(\mathbb{K}) \simeq QA_n(\mathbb{K})$ if and only if $P \approx \lambda Q$ for some $\lambda \in \mathbb{K} \setminus \{0\}$.*

Proof. If $P = \lambda RQR^T$ for some $R \in \text{GL}_n(\mathbb{K})$ and some $\lambda \in \mathbb{K} \setminus \{0\}$, then

$$PA_n(\mathbb{K}) = RQR^T A_n(\mathbb{K}) = R(QR^T A_n(\mathbb{K})R)R^{-1} = R(QA_n(\mathbb{K}))R^{-1}.$$

Conversely, assume that $PA_n(\mathbb{K}) = R(QA_n(\mathbb{K}))R^{-1}$ for some $R \in \text{GL}_n(\mathbb{K})$. Then the above computation yields $(RQR^T)^{-1}PA_n(\mathbb{K}) = A_n(\mathbb{K})$ hence Lemma 12 yields a non-zero scalar λ such that $(RQR^T)^{-1}P = \lambda I_n$. Therefore $P = R(\lambda Q)R^T$. \square

Remark 2 (A crucial remark). Let E be a finite dimensional vector space and b a (possibly non-symmetric) bilinear form on E such that $\forall x \in E \setminus \{0\}, b(x, x) \neq 0$. Given a non-zero vector $x \in E$, the hyperplane $H := \{y \in E : b(x, y) = 0\}$ is then a complementary subspace of $\text{span}(x)$ in E . By induction on the dimension of spaces, it follows that there exists a basis (f_1, \dots, f_n) of E which is *right-orthogonal* for b , i.e. $b(f_i, f_j) = 0$ for every $(i, j) \in \llbracket 1, n \rrbracket^2$ satisfying $i < j$. For a non-isotropic matrix $P \in \text{GL}_n(\mathbb{K})$, this may be interpreted as follows: P is congruent to a lower-triangular matrix T , and hence $PA_n(\mathbb{K})$ is similar to $TA_n(\mathbb{K})$. This remark will play a major part in our proof of Theorem 3.

Now, given non-isotropic matrices P and Q of $\text{GL}_n(\mathbb{K})$, we may examine when the two affine subspaces $I_n + PA_n(\mathbb{K})$ and $I_n + QA_n(\mathbb{K})$ are equivalent.

Proposition 14. *Let P and Q be non-isotropic matrices of $\text{GL}_n(\mathbb{K})$. Then $I_n + PA_n(\mathbb{K}) \sim I_n + QA_n(\mathbb{K})$ if and only if the quadratic forms $X \mapsto X^T P X$ and $X \mapsto X^T Q X$ are similar.*

Proof. • Assume first that $I_n + PA_n(\mathbb{K}) \sim I_n + QA_n(\mathbb{K})$, and choose a pair $(R, S) \in \text{GL}_n(\mathbb{K})^2$ such that $R(I_n + PA_n(\mathbb{K})) = (I_n + QA_n(\mathbb{K}))S$. Obviously S belongs to $(I_n + QA_n(\mathbb{K}))S$, hence $S = R(I_n + PA)$ for some $A \in$

$A_n(\mathbb{K})$. By comparing the translation vector spaces of $R(I_n + P A_n(\mathbb{K}))$ and $(I_n + Q A_n(\mathbb{K}))S$, we also find that $RP A_n(\mathbb{K}) = Q A_n(\mathbb{K})S = Q(S^T)^{-1} A_n(\mathbb{K})$. Therefore Proposition 13 yields a non-zero scalar λ such that $RP = \lambda Q(S^T)^{-1}$. It follows that $S^T = (I_n - AP^T)R^T$ and

$$\lambda Q = RPS^T = RP(I_n - AP^T)R^T = RPR^T - (RP)A(RP)^T.$$

Since A is alternate, we find that $\lambda X^T Q X = X^T (RPR^T) X = (R^T X)^T P (R^T X)$ for every $X \in \mathbb{K}^n$, and the quadratic forms $X \mapsto X^T Q X$ and $X \mapsto X^T P X$ are similar because R^T is non-singular.

- Conversely, assume that $X \mapsto X^T Q X$ and $X \mapsto X^T P X$ are similar. Then there is a non-singular matrix $R \in \text{GL}_n(\mathbb{K})$, a non-zero scalar λ and an alternate matrix A' such that $\lambda Q = RPR^T + A'$. The matrix $A := -(RP)^{-1}A'((RP)^T)^{-1}$ is congruent to $-A'$ and is therefore alternate. We set $S := R(I_n + PA)$. Note that $S = RP(P^{-1} + A)$ is non-singular: indeed, $\forall X \in \mathbb{K}^n \setminus \{0\}$, $X^T(P^{-1} + A)X = X^T P^{-1} X \neq 0$ since P^{-1} is non-isotropic, hence $P^{-1} + A$ is non-singular. However $S^T = (I_n - AP^T)R^T$, therefore

$$RPS^T = RPR^T - (RP)A(RP)^T = RPR^T + A' = \lambda Q.$$

We deduce that

$$R(P A_n(\mathbb{K})) = \lambda Q(S^T)^{-1} A_n(\mathbb{K}) = (Q A_n(\mathbb{K}))S.$$

We have just proven that the affine subspaces $R(I_n + P A_n(\mathbb{K}))$ and $(I_n + Q A_n(\mathbb{K}))S$ have S as common point and have the same translation vector space, hence they are equal. This yields $I_n + P A_n(\mathbb{K}) \sim I_n + Q A_n(\mathbb{K})$. \square

Finally, the following lemma will be a major key to unlock our proof of Theorem 3:

Lemma 15. *Let $n \geq 3$. Assume $\#\mathbb{K} \geq 3$. Let V be a $\binom{n}{2}$ -dimensional linear subspace of $M_n(\mathbb{K})$ which acts totally intransitively on \mathbb{K}^n . Assume that there exists a linear hyperplane H of V such that $H \subset A_n(\mathbb{K})$. Then $V = A_n(\mathbb{K})$.*

Proof. Let $A \in V$. We prove that A is alternate, i.e. that the quadratic form $q : X \mapsto X^T A X$ is zero. We denote by (e_1, \dots, e_n) the canonical basis of \mathbb{K}^n . Let $X \in \mathbb{K}^n \setminus \{0\}$. If $\dim(HX) = n - 1$ then $AX \in HX$ since $HX \subset VX \subsetneq \mathbb{K}^n$, and hence $q(X) = 0$.

If $\dim(HX) = n - 1$ for every $X \in \mathbb{K}^n \setminus \{0\}$, then we readily have $q = 0$.

Assume now that $\dim(HX_1) < n - 1$ for some $X_1 \in \mathbb{K}^n \setminus \{0\}$.

This shows that there exists $X_2 \in \mathbb{K}^n \setminus \text{span}(X_1)$ such that $X_2^T M X_1 = 0$ for every $M \in H$. Let $X_3 \in \mathbb{K}^n \setminus \text{span}(X_1, X_2)$. We may choose a non-singular matrix $P \in \text{GL}_n(\mathbb{K})$ such that $P e_i = X_i$ for every $i \in \llbracket 1, 3 \rrbracket$.

Then $V' := P^T V P$ acts totally intransitively on \mathbb{K}^n and contains the hyperplane $H' := P^T H P \subset \text{A}_n(\mathbb{K})$. We now have $e_2^T M e_1 = 0$ for every $M \in H'$, hence H' is included in the space V_1 of all alternate matrices $A = (a_{i,j})$ of $\text{M}_n(\mathbb{K})$ such that $a_{2,1} = 0$. The dimension of this space is obviously $\binom{n}{2} - 1$, and therefore $H' = V_1$. Then it is obvious that $\dim(H' e_3) = n - 1$ and hence $\dim(HX_3) = n - 1$.

We have therefore proven that

$$\forall X \in \mathbb{K}^n \setminus \text{span}(X_1, X_2), \quad q(X) = 0.$$

It now suffices to show that q vanishes everywhere on $\text{span}(X_1, X_2)$.

Let $X \in \text{span}(X_1, X_2) \setminus \{0\}$. We choose an arbitrary vector $X_3 \in \mathbb{K}^n \setminus \text{span}(X_1, X_2)$. The plane $\text{span}(X, X_3)$ satisfies $\text{span}(X, X_3) \cap \text{span}(X_1, X_2) = \text{span}(X)$. Since $\#\mathbb{K} > 2$, this plane has at least four distinct 1-dimensional subspaces, three of which are different from $\text{span}(X)$. We deduce that the quadratic form q vanishes on at least three 1-dimensional subspaces of $\text{span}(X, X_3)$. Classically, this shows that q vanishes everywhere on $\text{span}(X, X_3)$ (indeed, a non-zero homogeneous polynomial of degree 2 on \mathbb{K}^2 has at most 2 zeroes in the projective line $\mathbb{P}(\mathbb{K}^2)$). In particular $q(X) = 0$. We deduce that $q = 0$, which completes our proof. \square

3 The uniqueness statement in the two classification theorems

The uniqueness statement in Theorem 4 is equivalent to the following result, which we prove right away:

Proposition 16. *Let (P_1, \dots, P_p) and (Q_1, \dots, Q_q) be two families of non-isotropic matrices, respectively of $\text{GL}_{n_1}(\mathbb{K}) \times \dots \times \text{GL}_{n_p}(\mathbb{K})$ and $\text{GL}_{m_1}(\mathbb{K}) \times$*

$\cdots \times \mathrm{GL}_{m_q}(\mathbb{K})$. In order that

$$P_1 A_{n_1}(\mathbb{K}) \vee \cdots \vee P_p A_{n_p}(\mathbb{K}) \simeq Q_1 A_{m_1}(\mathbb{K}) \vee \cdots \vee Q_q A_{m_q}(\mathbb{K}),$$

it is necessary and sufficient that $q = p$ and P_k be congruent to a scalar multiple of Q_k for every $k \in \llbracket 1, p \rrbracket$.

Proof. The “sufficient condition” statement follows immediately from Proposition 13.

For the converse statement, set $V := P_1 A_{n_1}(\mathbb{K}) \vee \cdots \vee P_p A_{n_p}(\mathbb{K})$ and $W := Q_1 A_{m_1}(\mathbb{K}) \vee \cdots \vee Q_q A_{m_q}(\mathbb{K})$.

For $k \in \llbracket 1, p \rrbracket$, set $F_k := \mathbb{K}^{n_1 + \cdots + n_k} \times \{0\} \subset \mathbb{K}^n$, where $n := n_1 + \cdots + n_p$. Set also $F_0 = \{0\}$ and denote by (e_1, \dots, e_n) the canonical basis of \mathbb{K}^n . Set $k \in \llbracket 1, p \rrbracket$. Our key statement is the set of equalities:

$$\forall X \in F_k \setminus F_{k-1}, \quad \dim(VX) = n_1 + \cdots + n_k - 1.$$

Note first that the case $X = e_{n_1 + \cdots + n_{k-1} + 1}$ follows trivially from Lemma 9.

Consider now an arbitrary vector $X \in F_k \setminus F_{k-1}$. Then $e_1, \dots, e_{n_1 + \cdots + n_{k-1}}, X$ are linearly independent, and may therefore be completed as a basis $(e_1, \dots, e_{n_1 + \cdots + n_{k-1}}, X, f_2, \dots, f_{n_k})$ of F_k . Therefore

$$\mathcal{B} := (e_1, \dots, e_{n_1 + \cdots + n_{k-1}}, X, f_2, \dots, f_{n_k}, e_{n_1 + \cdots + n_k + 1}, \dots, e_n)$$

is a basis of \mathbb{K}^n and the matrix of coordinates R of \mathcal{B} in the canonical basis of \mathbb{K}^n belongs to $\mathrm{GL}_{n_1}(\mathbb{K}) \vee \cdots \vee \mathrm{GL}_{n_p}(\mathbb{K})$ and satisfies $Re_{n_1 + \cdots + n_{k-1} + 1} = X$. Proposition 13 thus yields a list of non-isotropic matrices $(P'_1, \dots, P'_p) \in \mathrm{GL}_{n_1}(\mathbb{K}) \times \cdots \times \mathrm{GL}_{n_p}(\mathbb{K})$ for which

$$RVR^{-1} \subset P'_1 A_{n_1}(\mathbb{K}) \vee \cdots \vee P'_p A_{n_p}(\mathbb{K})$$

and therefore $RVR^{-1} = P'_1 A_{n_1}(\mathbb{K}) \vee \cdots \vee P'_p A_{n_p}(\mathbb{K})$ as the dimensions equal $\binom{n}{2}$ on both sides. Applying the special case of $e_{n_1 + \cdots + n_{k-1} + 1}$ to RVR^{-1} then yields $\dim(VX) = \dim(RVX) = \dim(RVR^{-1})(RX) = n_1 + \cdots + n_k - 1$.

It follows that

$$\{\dim(VX) \mid X \in \mathbb{K}^n\} = \{0, n_1 - 1, n_1 + n_2 - 1, \dots, n_1 + \cdots + n_p - 1\}$$

has cardinality $p+1$. The same holds for W instead of V with the m_j 's in place of the n_k 's. Since V is similar to W , one has $\{\dim(VX) \mid X \in \mathbb{K}^n\} = \{\dim(WX) \mid$

$X \in \mathbb{K}^n\}$ and we deduce successively that $q = p$ and $(n_1, \dots, n_p) = (m_1, \dots, m_q)$. Now, set $P \in \text{GL}_n(\mathbb{K})$ such that $W = P^{-1}VP$. For every $k \in \llbracket 1, p \rrbracket$, remark that

$$\{X \in \mathbb{K}^n : \dim VX \leq n_1 + \dots + n_k - 1\} = F_k = \{X \in \mathbb{K}^n : \dim WX \leq n_1 + \dots + n_k - 1\},$$

hence P stabilizes F_k . This shows that $P \in \text{GL}_{n_1}(\mathbb{K}) \vee \dots \vee \text{GL}_{n_p}(\mathbb{K})$, which in turn proves that $P_k A_{n_k}(\mathbb{K})$ is similar to $Q_k A_{n_k}(\mathbb{K})$ for every $k \in \llbracket 1, p \rrbracket$. Proposition 13 finally yields that P_k is congruent to a scalar multiple of Q_k , for every $k \in \llbracket 1, p \rrbracket$. \square

Proposition 17. *Let (P_1, \dots, P_p) and (Q_1, \dots, Q_q) be two families of non-isotropic matrices, respectively in $\text{GL}_{n_1}(\mathbb{K}) \times \dots \times \text{GL}_{n_p}(\mathbb{K})$ and $\text{GL}_{m_1}(\mathbb{K}) \times \dots \times \text{GL}_{m_q}(\mathbb{K})$. In order that*

$$(I_{n_1} + P_1 A_{n_1}(\mathbb{K})) \vee \dots \vee (I_{n_p} + P_p A_{n_p}(\mathbb{K})) \sim (I_{m_1} + Q_1 A_{m_1}(\mathbb{K})) \vee \dots \vee (I_{m_q} + Q_q A_{m_q}(\mathbb{K})),$$

it is necessary and sufficient that $q = p$ and that the (non-isotropic) quadratic form $X \mapsto X^T P_k X$ be similar to $X \mapsto X^T Q_k X$ for every $k \in \llbracket 1, p \rrbracket$.

Proof. The “sufficient condition” statement follows trivially from Proposition 14. For the converse statement, let us set $\mathcal{V} := (I_{n_1} + P_1 A_{n_1}(\mathbb{K})) \vee \dots \vee (I_{n_p} + P_p A_{n_p}(\mathbb{K}))$ and $\mathcal{W} := (I_{m_1} + Q_1 A_{m_1}(\mathbb{K})) \vee \dots \vee (I_{m_q} + Q_q A_{m_q}(\mathbb{K}))$, and assume that $\mathcal{V} \sim \mathcal{W}$. Choose two non-singular matrices R and S such that $\mathcal{W} = R\mathcal{V}S$. Denote by V (resp. by W) the translation vector space of \mathcal{V} (resp. of \mathcal{W}), and set $n := \sum_{k=1}^p n_k$. Then

$$\mathcal{W} = (RS)S^{-1}(I_n + V)S = (RS)(I_n + S^{-1}VS).$$

In particular $RS \in \mathcal{W}$ and the comparison of translation vector spaces yields $S^{-1}VS = (RS)^{-1}W$. The first result yields that RS is upper block-triangular with diagonal blocks R_1, \dots, R_q where $R_k \in \text{GL}_{m_k}(\mathbb{K})$ for every $k \in \llbracket 1, q \rrbracket$. Thus

$$S^{-1}VS = (RS)^{-1}W = (R_1^{-1}Q_1) A_{m_1}(\mathbb{K}) \vee \dots \vee (R_q^{-1}Q_q) A_{m_q}(\mathbb{K})$$

and the $R_k^{-1}Q_k$ ’s are necessarily non-isotropic since $S^{-1}VS$ has a trivial spectrum. We deduce from Proposition 16 that $(n_1, \dots, n_p) = (m_1, \dots, m_q)$. With the line of reasoning from the proof of Proposition 16, we also find that $S \in$

$\mathrm{GL}_{n_1}(\mathbb{K}) \vee \cdots \vee \mathrm{GL}_{n_p}(\mathbb{K})$. However we already know that RS belongs to $\mathrm{GL}_{n_1}(\mathbb{K}) \vee \cdots \vee \mathrm{GL}_{n_p}(\mathbb{K})$ and hence $R = (RS)S^{-1} \in \mathrm{GL}_{n_1}(\mathbb{K}) \vee \cdots \vee \mathrm{GL}_{n_p}(\mathbb{K})$.

Returning to $RVS = \mathcal{W}$ finally entails that $I_{n_k} + Q_k A_{n_k}(\mathbb{K})$ is equivalent to $I_{n_k} + P_k A_{n_k}(\mathbb{K})$ for each $k \in \llbracket 1, p \rrbracket$, and Proposition 14 then yields that $X \mapsto X^T P_k X$ is similar to $X \mapsto X^T Q_k X$ for each $k \in \llbracket 1, p \rrbracket$. \square

4 Structure of the irreducible maximal spaces with a trivial spectrum

In the whole section, we assume $\#\mathbb{K} \geq 3$. We will prove Theorem 3 by induction. The case $n = 1$ needs no explanation.

4.1 The case $n = 2$

Let V be an irreducible maximal linear subspace of $M_2(\mathbb{K})$ with a trivial spectrum. Then $V = \mathrm{span}(M)$ for some $M \in M_2(\mathbb{K}) \setminus \{0\}$ with no non-zero eigenvalue. If 0 is an eigenvalue of M , then M is triangularizable and V is not irreducible.

Hence M is non-singular. Setting $K := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $P := MK^{-1}$, we readily have $PA_2(\mathbb{K}) = \mathrm{span}(M) = V$ and Lemma 10 shows that P is non-isotropic.

4.2 Setting things up

Let $n \geq 2$ and assume that the result of Theorem 3 holds for any positive integer $k \leq n$. Let $V \subset M_{n+1}(\mathbb{K})$ be a maximal subspace with a trivial spectrum. Denote by (e_1, \dots, e_{n+1}) the canonical basis of \mathbb{K}^{n+1} . We wish to show that V is reducible or similar to $PA_{n+1}(\mathbb{K})$ for some $P \in \mathrm{GL}_{n+1}(\mathbb{K})$, in which case Lemma 10 guarantees that P must be non-isotropic.

Of course, this amounts to finding a basis of \mathbb{K}^{n+1} in which all the endomorphisms $X \mapsto MX$ of \mathbb{K}^{n+1} , for $M \in V$, have a “reduced” shape that is essentially the one described in Theorem 4. The first problem is how to select the last vector f_{n+1} of such a basis. Since the rank of an alternate matrix is even, an obvious necessary condition is that V should not contain any matrix with $\mathrm{span}(f_{n+1})$ as column space. Our starting point is that such a vector exists (and may even be chosen amongst the canonical basis of \mathbb{K}^{n+1}). This has already been proven in [10, Proposition 10]: we reproduce a proof since it is short and the result is crucial to our study.

Lemma 18. *Let W be a linear subspace of $M_p(\mathbb{K})$ with a trivial spectrum. Then there exists a non-zero vector $X \in \mathbb{K}^p$ such that W contains no matrix M with $\text{span}(X)$ as column space.*

Proof. Denote by (e_1, \dots, e_p) the canonical basis of \mathbb{K}^p . For $X \in \mathbb{K}^p \setminus \{0\}$, set $W_X := \{M \in W : \text{Im}(M) \subset \text{span}(X)\}$. For $(i, j) \in \llbracket 1, p \rrbracket^2$, denote by $E_{i,j}$ the matrix of $M_p(\mathbb{K})$ with zero entries everywhere except at the (i, j) -spot where the entry is 1.

We prove, by induction on p , that there exists an index $i \in \llbracket 1, p \rrbracket$ such that $W_{e_i} = \{0\}$. The case $p = 1$ is trivial.

Assume that $W_{e_i} \neq \{0\}$ for every $i \in \llbracket 1, p \rrbracket$, denote by W' the linear subspace of W consisting of its matrices with zero as last row, and write every $M \in W'$ as

$$M = \begin{bmatrix} J(M) & [?]_{(p-1) \times 1} \\ [0]_{1 \times (p-1)} & 0 \end{bmatrix} \quad \text{with } J(M) \in M_{p-1}(\mathbb{K}).$$

Then $J(W')$ is a linear subspace of $M_{p-1}(\mathbb{K})$ with a trivial spectrum. The induction hypothesis yields an index $i \in \llbracket 1, p-1 \rrbracket$ such that $J(W')_{e_i} = \{0\}$.

Since $W_{e_i} \neq \{0\}$, we find a matrix $M \in W$ such that $\text{Im}(M) = \text{span}(e_i)$. Then $M \in W'$ and it follows from $J(W')_{e_i} = \{0\}$ that M is a non-zero scalar multiple of $E_{i,p}$. Therefore $E_{i,p} \in W$.

Now, taking an arbitrary permutation matrix $P \in \text{GL}_n(\mathbb{K})$ and applying the previous step to PWP^{-1} yields the following generalization: for every $j \in \llbracket 1, p \rrbracket$, there exists an integer $f(j) \in \llbracket 1, p \rrbracket \setminus \{j\}$ such that $E_{f(j),j} \in W$.

We choose a *cycle* for the map $f : \llbracket 1, p \rrbracket \rightarrow \llbracket 1, p \rrbracket$, i.e. a list (j_1, \dots, j_r) of distinct elements of $\llbracket 1, p \rrbracket$ such that $f(j_1) = j_2, \dots, f(j_{r-1}) = j_r$ and $f(j_r) = j_1$. The matrix $A := \sum_{k=1}^r E_{f(j_k), j_k}$ then belongs to W although 1 is an eigenvalue of it (a

corresponding eigenvector being $\sum_{k=1}^r e_{j_k}$). This is a contradiction, which shows that $W_{e_i} = \{0\}$ for some $i \in \llbracket 1, p \rrbracket$. \square

By conjugating V with an appropriate invertible matrix, we then lose no generality assuming that no matrix of V has $\text{span}(e_{n+1})$ as column space and that $Ve_{n+1} \subset \text{span}(e_1, \dots, e_n)$ (since $e_{n+1} \notin Ve_{n+1}$). This means that every matrix of V has a 0 entry at the $(n+1, n+1)$ -spot.

In order to complete the choice of a “good” basis for V , we now turn to the first n vectors f_1, \dots, f_n . The basic idea is to find the projections of f_1, \dots, f_n

onto $\text{span}(e_1, \dots, e_n)$ and alongside $\text{span}(e_{n+1})$ by applying the induction hypothesis to a subspace of $M_n(\mathbb{K})$ that is deduced from V (the space V_{ul} defined below), and then apply the induction hypothesis once more to find the projections of f_1, \dots, f_n onto $\text{span}(e_{n+1})$ alongside $\text{span}(e_1, \dots, e_n)$.

Consider the subspace W of V consisting of its matrices with zero as last column. For $M \in W$, write

$$M = \begin{bmatrix} K(M) & [0]_{n \times 1} \\ L(M) & 0 \end{bmatrix} \quad \text{with } K(M) \in M_n(\mathbb{K}) \text{ and } L(M) \in M_{1,n}(\mathbb{K}),$$

and set

$$V_{ul} := K(W)$$

(the subscript “ ul ” stands for “upper left”). The rank theorem shows that

$$\dim V = \dim W + \dim(Ve_{n+1}) \quad \text{and} \quad \dim W = \dim \text{Ker } K + \dim V_{ul}.$$

However, our assumptions mean that $\text{Ker } K = \{0\}$, hence

$$\dim V = \dim V_{ul} + \dim(Ve_{n+1}).$$

Obviously, V_{ul} is a linear subspace of $M_n(\mathbb{K})$ with a trivial spectrum hence $\dim V_{ul} \leq \binom{n}{2}$. Moreover $\dim(Ve_{n+1}) \leq n$ since V acts totally intransitively on \mathbb{K}^{n+1} . We deduce that

$$\binom{n+1}{2} = \dim V = \dim V_{ul} + \dim(Ve_{n+1}) \leq \binom{n}{2} + n = \binom{n+1}{2},$$

hence

$$\dim V_{ul} = \binom{n}{2} \quad \text{and} \quad \dim(Ve_{n+1}) = n.$$

In this reduced situation, we conclude that:

1. V_{ul} is a maximal linear subspace of $M_n(\mathbb{K})$ with a trivial spectrum.
2. $Ve_{n+1} = \text{span}(e_1, \dots, e_n)$.

Applying the induction hypothesis to V_{ul} together with Remark 2 shows that we may find non-isotropic *lower-triangular* matrices P_1, \dots, P_r such that

$$V_{ul} \simeq P_1 A_{n_1}(\mathbb{K}) \vee \dots \vee P_r A_{n_r}(\mathbb{K}).$$

This shows that, by conjugating V with a well-chosen matrix of the form $\begin{bmatrix} R & [0]_{n \times 1} \\ [0]_{1 \times n} & 1 \end{bmatrix}$ for some $R \in \text{GL}_n(\mathbb{K})$, we lose no generality assuming that

$$V_{ul} = P_1 A_{n_1}(\mathbb{K}) \vee \cdots \vee P_r A_{n_r}(\mathbb{K}) \quad \text{and} \quad P_1 = \begin{bmatrix} 1 & [0]_{1 \times (n_1-1)} \\ C'_1 & P'_1 \end{bmatrix}$$

for some lower-triangular matrix $P'_1 \in \text{M}_{n_1-1}(\mathbb{K})$ (possibly of size 0) and some column matrix $C'_1 \in \text{M}_{n_1-1,1}(\mathbb{K})$.

Remark 3 (An important remark on block-diagrams). *From now on, and unless specified otherwise, every matrix M of V will be systematically seen with the following 3×3 block decomposition:*

$$M = \begin{bmatrix} ? & [?]_{1 \times (n-1)} & ? \\ [?]_{(n-1) \times 1} & [?]_{n-1} & [?]_{(n-1) \times 1} \\ ? & [?]_{1 \times (n-1)} & ? \end{bmatrix}$$

i.e. the four question marks represent single entries, whilst the others represent submatrices with sizes as indicated by the subscript (where the central subscript $n-1$ denotes a $(n-1) \times (n-1)$ block).

If $n_1 > 1$, we set $s := r$, $(i_1, \dots, i_s) := (n_1 - 1, n_2, \dots, n_r)$ and $(R_1, \dots, R_s) := (P'_1, P_2, \dots, P_r)$.

If $n_1 = 1$, we set $s := r - 1$, $(i_1, \dots, i_s) := (n_2, \dots, n_r)$ and $(R_1, \dots, R_s) := (P_2, \dots, P_r)$.

In any case, we set

$$V_m := R_1 A_{i_1}(\mathbb{K}) \vee \cdots \vee R_s A_{i_s}(\mathbb{K})$$

(the subscript “ m ” stands for “middle”). Here are two consequences of the above reductions (with the block decompositions laid out in Remark 3):

- (i) For every $L \in \text{M}_{1,n-1}(\mathbb{K})$, the subspace V contains a matrix of the form

$$\begin{bmatrix} ? & L & 0 \\ ? & ? & 0 \\ ? & ? & 0 \end{bmatrix};$$

- (ii) For every $U \in V_m$, the subspace V contains a matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ ? & ? & 0 \end{bmatrix}.$$

Proof of statement (i). Let $L_1 \in M_{1,n_1-1}(\mathbb{K})$. Then

$$P_1 \times \begin{bmatrix} 0 & L_1 \\ -L_1^T & [0]_{n_1-1} \end{bmatrix} = \begin{bmatrix} ? & L_1 \\ [?]_{(n_1-1) \times 1} & [?]_{n_1-1} \end{bmatrix}$$

and $\begin{bmatrix} 0 & L_1 \\ -L_1^T & [0]_{n_1-1} \end{bmatrix}$ is alternate. Since $V_{ul} = P_1 A_{n_1}(\mathbb{K}) \vee \dots \vee P_r A_{n_r}(\mathbb{K})$, we deduce that, for every $L \in M_{1,n_1-1}(\mathbb{K})$, the subspace V_{ul} contains a matrix of the form $\begin{bmatrix} ? & L \\ [?]_{(n_1-1) \times 1} & [?]_{n_1-1} \end{bmatrix}$, and the conclusion follows from the definition of V_{ul} . \square

Proof of statement (ii). We will only tackle the case $n_1 > 1$, the case $n_1 = 1$ being essentially similar (and even simpler). For every $M \in P_2 A_{n_2}(\mathbb{K}) \vee \dots \vee P_r A_{n_r}(\mathbb{K})$ and every $N \in M_{n_1-1,n-n_1}(\mathbb{K})$, we know that V_{ul} contains the matrix

$$\begin{bmatrix} 0 & [0]_{1 \times (n_1-1)} & [0]_{1 \times (n-n_1)} \\ [0]_{(n_1-1) \times 1} & [0]_{n_1-1} & N \\ [0]_{(n-n_1) \times 1} & [0]_{(n-n_1) \times (n_1-1)} & M \end{bmatrix}. \text{ Let } A \in A_{n_1-1}(\mathbb{K}). \text{ Then}$$

$$P_1 \times \begin{bmatrix} 0 & [0]_{1 \times (n_1-1)} \\ [0]_{(n_1-1) \times 1} & A \end{bmatrix} = \begin{bmatrix} 0 & [0]_{1 \times (n_1-1)} \\ [0]_{(n_1-1) \times 1} & P_1' A \end{bmatrix}$$

and it follows that V_{ul} contains a matrix of the form

$$\begin{bmatrix} 0 & [0]_{1 \times (n_1-1)} & [0]_{1 \times (n-n_1)} \\ [0]_{(n_1-1) \times 1} & P_1' A & [0]_{(n_1-1) \times (n-n_1)} \\ [0]_{(n-n_1) \times 1} & [0]_{(n-n_1) \times (n_1-1)} & [0]_{(n-n_1) \times (n-n_1)} \end{bmatrix}.$$

With the respective definitions of V_m and V_{ul} , point (ii) follows easily. \square

Let now $C \in M_{n-1,1}(\mathbb{K})$. Since $V e_{n+1} = \text{span}(e_1, \dots, e_n)$, we know that V contains a matrix of the form

$$\begin{bmatrix} ? & ? & 0 \\ ? & ? & C \\ ? & ? & 0 \end{bmatrix}.$$

Adding an appropriate matrix given by statement (i), and remembering that 0 is the only possible eigenvalue for a matrix in V , we deduce:

(iii) V contains a matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ ? & ? & C \\ ? & ? & 0 \end{bmatrix}.$$

Denote now by V' the subspace of V consisting of its matrices with zero as first row. For $M \in V'$, write

$$M = \begin{bmatrix} 0 & [0]_{1 \times n} \\ [?]_{n \times 1} & \mathcal{J}(M) \end{bmatrix} \quad \text{with } \mathcal{J}(M) \in M_n(\mathbb{K}),$$

and set

$$V_{lr} := \mathcal{J}(V')$$

(the subscript “ lr ” stands for “lower right”). Note that the subspace V_{lr} of $M_n(\mathbb{K})$ has a trivial spectrum and that it contains:

- (a) A matrix of the form $\begin{bmatrix} U & [0]_{(n-1) \times 1} \\ [?]_{1 \times (n-1)} & 0 \end{bmatrix}$ for every $U \in V_m$ (by statement (ii));
- (b) A matrix of the form $\begin{bmatrix} [?]_{n-1} & C \\ [?]_{1 \times (n-1)} & 0 \end{bmatrix}$ for every $C \in M_{n-1,1}(\mathbb{K})$ (by statement (iii)).

Since $\dim V_m = \binom{n-1}{2}$, we deduce that $\dim V_{lr} \geq \binom{n-1}{2} + (n-1) = \binom{n}{2}$. However $\dim V_{lr} \leq \binom{n}{2}$ since V_{lr} has a trivial spectrum. It thus follows from statements (a) and (b) that:

- (c) V_{lr} contains, for every $U \in V_m$, a *unique* matrix of the form $\begin{bmatrix} U & [0]_{(n-1) \times 1} \\ ? & 0 \end{bmatrix}$;
- (d) Every matrix of V_{lr} with zero as last column has the form $\begin{bmatrix} U & [0]_{(n-1) \times 1} \\ ? & 0 \end{bmatrix}$ for some $U \in V_m$.

A key point now is that V_{lr} is a maximal linear subspace of $M_n(\mathbb{K})$ with a trivial spectrum. One may thus be tempted to apply the induction hypothesis to V_{lr} . However, the problem is that using a new change of basis blindly risks destroying the previous reduced form of V_{ul} ! As we shall now see, the fact that V_m is already reduced forces V_{lr} to be already in the reduced form of Theorem 4 (i.e. no further change of basis is necessary at this point).

Claim 1. *The subspace V_{lr} has a “roughly reduced” shape i.e. there exists an integer $q \geq 1$, a non-isotropic matrix $Q \in \text{GL}_q(\mathbb{K})$ and a maximal subspace W of $M_{n-q}(\mathbb{K})$ with a trivial spectrum such that*

$$V_{lr} = W \vee Q A_q(\mathbb{K}).$$

Proof. Applying the induction hypothesis to V_{lr} , we recover a matrix $P \in \text{GL}_n(\mathbb{K})$, a non-isotropic matrix $Q' \in \text{GL}_q(\mathbb{K})$ (possibly with $q = n$) and a maximal subspace W' of $M_{n-q}(\mathbb{K})$ with a trivial spectrum such that

$$PV_{lr}P^{-1} = W' \vee Q' A_q(\mathbb{K}).$$

Note, using statement (b), that $\dim(V_{lr}e_n) = n - 1$ whereas $\dim(PV_{lr}P^{-1}x) < n - 1$ for every $x \in \text{span}(e_1, \dots, e_{n-q})$ (since W' acts totally intransitively on \mathbb{K}^{n-q}). Hence $Pe_n \notin \text{span}(e_1, \dots, e_{n-q})$. Multiplying P with a well-chosen matrix of $\text{GL}_{n-q}(\mathbb{K}) \vee \text{GL}_q(\mathbb{K})$, we lose no generality assuming that $Pe_n = e_n$.

Assume first that $q = 1$. Then $V_{lr}e_n = \text{span}(e_1, \dots, e_{n-1}) = (PV_{lr}P^{-1})e_n$ whilst $PV_{lr}P^{-1}e_n = P(V_{lr}e_n)$, which shows that P stabilizes $\text{span}(e_1, \dots, e_{n-1})$. Therefore $P \in \text{GL}_{n-1}(\mathbb{K}) \vee \{1\}$ and $V_{lr} = W \vee A_1(\mathbb{K})$ for some maximal linear subspace W of $M_{n-1}(\mathbb{K})$ with a trivial spectrum.

Assume, for the rest of the proof, that $q > 1$. Our aim is to prove that $P \in \text{GL}_{n-q}(\mathbb{K}) \vee \text{GL}_q(\mathbb{K})$, and it will follow that $V_{lr} = W \vee Q A_q(\mathbb{K})$ for some maximal linear subspace W of $M_{n-q}(\mathbb{K})$ with a trivial spectrum and some non-isotropic matrix $Q \in \text{GL}_q(\mathbb{K})$.

Set

$$H := \{M \in V_{lr} : Me_n = 0\}$$

i.e. H is the set of all matrices of V_{lr} with 0 as last column. Notice that $PHP^{-1} = \{M \in PV_{lr}P^{-1} : Me_n = 0\}$ since $Pe_n = e_n$. Notice also that

$$\text{span}(e_1, \dots, e_{n-1-i_s}) \subset \text{span}(e_1, \dots, e_{n-1}) = V_{lr}e_n$$

(this uses statement (b) and the fact that $V_{lr}e_n \neq \mathbb{K}^n$) and that

$$\text{span}(e_1, \dots, e_{n-q}) \subset (PV_{lr}P^{-1})e_n.$$

- *Case 1: $i_s > 1$.*

– We first claim that

$$\forall x \in V_{lr}e_n, \quad \dim Hx < n - 2 \Leftrightarrow x \in \text{span}(e_1, \dots, e_{n-1-i_s}). \quad (1)$$

Indeed, let $x \in \text{span}(e_1, \dots, e_{n-1})$ seen as a vector of \mathbb{K}^{n-1} with the canonical identification $\mathbb{K}^{n-1} \simeq \mathbb{K}^{n-1} \times \{0\} \subset \mathbb{K}^n$. By statements (c) and (d), one has

$$\dim V_mx \leq \dim Hx \leq 1 + \dim V_mx.$$

If $x \in \text{span}(e_1, \dots, e_{n-1-i_s})$, then the line of reasoning from the proof of Proposition 16 yields $\dim V_mx \leq n - i_s - 2$ and hence $\dim Hx \leq n - i_s - 1 < n - 2$; otherwise $\dim V_mx = n - 2$ and hence $\dim Hx \geq n - 2$.

– Moreover, we claim that

$$\forall x \in (PV_{lr}P^{-1})e_n, \quad \dim(PHP^{-1}x) < n-2 \Leftrightarrow x \in \text{span}(e_1, \dots, e_{n-q}). \quad (2)$$

The implication \Leftarrow follows from $PV_{lr}P^{-1} = W' \vee Q' A_q(\mathbb{K})$ since W' acts totally intransitively on \mathbb{K}^{n-q} and $q > 1$.

For the converse implication, notice first that the equality $PV_{lr}P^{-1} = W' \vee Q' A_q(\mathbb{K})$ yields $(PV_{lr}P^{-1})e_n = \text{span}(e_1, \dots, e_{n-q}) \oplus G$ for some $(q-1)$ -dimensional subspace G of $\text{span}(e_{n-q+1}, \dots, e_n)$ which does not contain e_n (note that $(PV_{lr}P^{-1})e_n$ cannot contain e_n since $PV_{lr}P^{-1}$ has a trivial spectrum). Consider a vector $x \in G \setminus \{0\}$. The subspace PHP^{-1} contains, for every $A \in A_{q-1}(\mathbb{K})$, and every $B \in M_{n-q,q}(\mathbb{K})$ with zero as last column, the matrix

$$\begin{bmatrix} [0]_{n-q} & B \\ [0]_{q \times (n-q)} & C \end{bmatrix} \quad \text{where } C = Q' \times \begin{bmatrix} A & [0]_{(q-1) \times 1} \\ [0]_{1 \times (q-1)} & 0 \end{bmatrix}.$$

Since x belongs to $\text{span}(e_{n-q+1}, \dots, e_n)$ and is linearly independent from e_n , it easily follows that $\dim(PHP^{-1})x \geq n - 2$.

Let now $x \in (PV_{lr}P^{-1})e_n \setminus \text{span}(e_1, \dots, e_{n-q})$. Then we have a decomposition $x = z + y$ with $z \in \text{span}(e_1, \dots, e_{n-q})$ and $y \in G \setminus \{0\}$. Obviously, there exists a non-singular matrix $R \in \{I_{n-q}\} \vee \{I_q\}$ such that $Rx = y$. Replacing P with RP , we thus reduce the situation to the one where $x \in G \setminus \{0\}$, which we have treated before. Implication \Rightarrow in statement (2) follows.

Since $X \mapsto PX$ is linear, $Pe_n = e_n$, $\text{span}(e_1, \dots, e_{n-q}) \subset (PV_{lr}P^{-1})e_n$, and $\text{span}(e_1, \dots, e_{n-1-i_s}) \subset V_{lr}e_n$, we deduce from statements (1) and (2) that $X \mapsto PX$ induces an isomorphism from $\text{span}(e_1, \dots, e_{n-1-i_s})$ to $\text{span}(e_1, \dots, e_{n-q})$, hence $i_s = q - 1$ and $P \in \text{GL}_{n-q}(\mathbb{K}) \vee \text{GL}_q(\mathbb{K})$.

• *Case 2: $i_s = 1$.*

– Notice first that $\text{span}(e_1, \dots, e_{n-1}) = V_{lr}e_n$ and

$$\forall x \in V_{lr}e_n, \dim(Hx \cap V_{lr}e_n) < n - 2 \quad \text{if } x \in \text{span}(e_1, \dots, e_{n-2}). \quad (3)$$

Indeed, for every $x \in \text{span}(e_1, \dots, e_{n-2})$, statements (c) and (d) show that $\dim(Hx \cap V_{lr}e_n) \leq \dim(V_mx)$ (where x is naturally seen as a vector of \mathbb{K}^{n-1}), and the definition of V_m shows, since $i_s = 1$, that $\dim(V_mx) < n - 2$.

– On the other hand, we claim that

$$\forall x \in (PV_{lr}P^{-1})e_n, \dim((PHP^{-1})x \cap (PV_{lr}P^{-1})e_n) < n - 2 \Leftrightarrow x \in \text{span}(e_1, \dots, e_{n-q}). \quad (4)$$

Indeed, for any $x \in \text{span}(e_1, \dots, e_{n-q})$, one has

$$\dim((PHP^{-1})x \cap (PV_{lr}P^{-1})e_n) \leq \dim(PV_{lr}P^{-1})x \leq n - q - 1 < n - 2.$$

Conversely, let $x \in (PV_{lr}P^{-1})e_n \setminus \text{span}(e_1, \dots, e_{n-q})$. Note first that $(PHP^{-1})x \subset (PV_{lr}P^{-1})e_n$. In order to see this, we naturally identify \mathbb{K}^n with $\mathbb{K}^{n-q} \oplus \mathbb{K}^q$: the identity $PV_{lr}P^{-1} = W' \vee Q' A_q(\mathbb{K})$ yields $(PV_{lr}P^{-1})e_n = \mathbb{K}^{n-q} \times [Q'(\mathbb{K}^{q-1} \times \{0\})]$ whilst, for every $M \in PHP^{-1}$, the column space of M is included in $\mathbb{K}^{n-q} \times Q' \text{Im}(N)$, where $N = \begin{bmatrix} A_M & [0]_{(q-1) \times 1} \\ [0]_{1 \times (q-1)} & 0 \end{bmatrix}$ for some $A_M \in A_{q-1}(\mathbb{K})$; this shows that $\text{Im } M \subset (PV_{lr}P^{-1})e_n$ for every $M \in PHP^{-1}$.

With the same arguments as in the proof of statement (2), one may prove that $\dim(PHP^{-1})x = n - 2$, and hence $\dim((PHP^{-1})x \cap (PV_{lr}P^{-1})e_n) = \dim(PHP^{-1})x = n - 2$. Therefore statement (4) is established.

From statements (3) and (4), we deduce that the linear injection $X \mapsto PX$ maps $\text{span}(e_1, \dots, e_{n-2})$ into $\text{span}(e_1, \dots, e_{n-q})$, which shows that $q = 2$, $i_s = 1 = q - 1$ and $P \in \text{GL}_{n-q}(\mathbb{K}) \vee \text{GL}_q(\mathbb{K})$. This finishes our proof.

□

Now that we know that V_{lr} is “roughly reduced”, we may use the shape of V_m to better grasp the one of V_{lr} .

Take W , q and Q as in Claim 1. If $q = 1$, then obviously $W = V_m$.

Assume now that $q > 1$ and split $Q = \begin{bmatrix} Q_1 & [?]_{(q-1) \times 1} \\ [?]_{1 \times (q-1)} & ? \end{bmatrix}$ with $Q_1 \in M_{q-1}(\mathbb{K})$. Then Q_1 is still non-isotropic and statement (d) shows that V_m contains $W \vee Q_1 A_{q-1}(\mathbb{K})$, and hence $V_m = W \vee Q_1 A_{q-1}(\mathbb{K})$ since the dimensions are equal on both sides. By applying the induction hypothesis to W and by using the same arguments as in the proof of Proposition 16, we deduce that $W = R_1 A_{i_1}(\mathbb{K}) \vee \cdots \vee R_{s-1} A_{i_{s-1}}(\mathbb{K})$ and $Q_1 A_{q-1}(\mathbb{K}) = R_s A_{i_s}(\mathbb{K})$.

Therefore

$$V_{lr} = \begin{cases} R_1 A_{i_1}(\mathbb{K}) \vee \cdots \vee R_s A_{i_s}(\mathbb{K}) \vee A_1(\mathbb{K}) & \text{if } q = 1 \\ R_1 A_{i_1}(\mathbb{K}) \vee \cdots \vee R_{s-1} A_{i_{s-1}}(\mathbb{K}) \vee Q A_q(\mathbb{K}) & \text{if } q > 1. \end{cases}$$

Assume again that $q > 1$. Then Q need not be lower-triangular, so we have to reduce the situation a little further.

Since $V e_{n+1} = \text{span}(e_1, \dots, e_n)$, we find that $Q A_q(\mathbb{K}) e_q = \text{span}(e_1, \dots, e_{q-1})$ which shows that Q stabilizes $\text{span}(e_1, \dots, e_{q-1})$, i.e. $Q = \begin{bmatrix} T_0 & [?]_{(q-1) \times 1} \\ [0]_{1 \times (q-1)} & \alpha \end{bmatrix}$ for some $T_0 \in \text{GL}_{q-1}(\mathbb{K})$ and some $\alpha \in \mathbb{K} \setminus \{0\}$.

Note, since Q is non-singular, that a matrix of the form $M = QA$, with $A \in A_q(\mathbb{K})$, has zero as last column if and only if A has zero as last column. It then follows from the shape of V_m that $T_0 A_{q-1}(\mathbb{K}) = R_s A_{i_s}(\mathbb{K})$. Therefore $(R_s^{-1} T_0) A_{q-1}(\mathbb{K}) = A_{q-1}(\mathbb{K})$, and we deduce from Lemma 12 that T_0 is a scalar multiple of R_s . Since we may replace Q with a scalar multiple of itself, we lose no generality assuming that $T_0 = R_s$.

Finally we define $T_1 := \begin{bmatrix} I_{q-1} & C' \\ [0]_{1 \times (q-1)} & 1 \end{bmatrix} \in \text{GL}_q(\mathbb{K})$, where $C' := \frac{C}{\alpha}$, so that

$Q' := (T_1)^T Q T_1$ is lower-triangular; we replace V with $R V R^{-1}$ for $R := \begin{bmatrix} I_{n+1-q} & [0]_{(n+1-q) \times q} \\ [0]_{q \times (n+1-q)} & T_1^T \end{bmatrix}$.

For the sake of convenience (and symmetry), we now set

$$P := P_1 \quad \text{and} \quad p := n_1.$$

Let us see how the situation looks like after all those reductions:

(i) We still have

$$V_{ul} = P A_p(\mathbb{K}) \vee P_2 A_{n_2}(\mathbb{K}) \vee \cdots \vee P_r A_{n_r}(\mathbb{K})$$

and

$$V_m = R_1 A_{i_1}(\mathbb{K}) \vee \cdots \vee R_s A_{i_s}(\mathbb{K})$$

with the above notations (nothing has changed there).

Recall that $(i_1, \dots, i_s) = (n_2, \dots, n_r)$ if $p = 1$, otherwise $(i_1, \dots, i_s) = (n_1 - 1, n_2, \dots, n_r)$.

(ii) Either $q = 1$ and then

$$V_{lr} = R_1 A_{i_1}(\mathbb{K}) \vee \cdots \vee R_s A_{i_s}(\mathbb{K}) \vee Q A_1(\mathbb{K}) \quad \text{with } Q = 1,$$

or $q > 1$ and then

$$V_{lr} = R_1 A_{i_1}(\mathbb{K}) \vee \cdots \vee R_{s-1} A_{i_{s-1}}(\mathbb{K}) \vee Q A_q(\mathbb{K})$$

and

$$Q = \begin{bmatrix} R_s & [0]_{i_s \times 1} \\ L_1 & \alpha \end{bmatrix} \quad \text{with } \alpha \in \mathbb{K} \setminus \{0\} \text{ and } L_1 \in M_{1, q-1}(\mathbb{K}).$$

We set $\alpha := 1$ if $q = 1$.

(iii) Recall finally that if $p > 1$, then $P = \begin{bmatrix} 1 & [0]_{1 \times (p-1)} \\ C_1 & R_1 \end{bmatrix}$ for some $C_1 \in M_{p-1, 1}(\mathbb{K})$.

(iv) No matrix of V has $\text{span}(e_{n+1})$ as column space (no change there).

However, one important thing has changed: if $q > 1$, we no longer have $V e_{n+1} = \text{span}(e_1, \dots, e_n)$, rather $V e_{n+1} = \text{span}(e_1, \dots, e_{n+1-q}) \oplus H$ for some linear hyperplane H of $\text{span}(e_{n+2-q}, \dots, e_{n+1})$ which does not contain e_{n+1} . We still have $e_1 \in V e_{n+1}$, nevertheless. Set finally

$$Z := \begin{bmatrix} R_1 & & (0) \\ & \ddots & \\ (0) & & R_s \end{bmatrix} \in \text{GL}_{n-1}(\mathbb{K}).$$

From there, V will remain essentially fixed. We will prove separately:

- That the case $p = n = q$ (i.e. V_{ul} and V_{lr} are **glued**) leads to the equivalence of V with $A_{n+1}(\mathbb{K})$;
- That the case $p \neq n$ or $q \neq n$ (i.e. V_{ul} and V_{lr} are **unglued**) leads to the reducibility of V .

Prior to studying the two cases separately, we continue with general considerations that apply to both of them.

4.3 Special types of matrices in V

With the matrices L_1 and C_1 from the previous paragraph², set

$$\widetilde{L}_1 := \begin{bmatrix} [0]_{1 \times (n-q)} & L_1 \end{bmatrix} \in M_{1,n-1}(\mathbb{K}) \quad \text{and} \quad \widetilde{C}_1 := \begin{bmatrix} C_1 \\ [0]_{(n-p) \times 1} \end{bmatrix} \in M_{n-1,1}(\mathbb{K}).$$

Notation 7. For an arbitrary $L \in M_{1,n-1}(\mathbb{K})$, we define \overline{L} as the matrix of $M_{1,n-1}(\mathbb{K})$ with the same first $p-1$ entries as L and all the other ones equal to zero.

For an arbitrary $C \in M_{n-1,1}(\mathbb{K})$, we define \overline{C} as the matrix of $M_{n-1,1}(\mathbb{K})$ with the same last $q-1$ entries as C and all the other ones equal to zero.

Using the respective shapes of V_{ul} , V_m and V_{lr} , we now find important classes of matrices in V , together with an isolated matrix. First of all, taking arbitrary row matrices $L_0 \in M_{1,p-1}(\mathbb{K})$ and $L'_0 \in M_{1,n-p}(\mathbb{K})$, we know that V_{ul} contains a matrix of the form $\begin{bmatrix} PA & N \\ [0]_{(n-p) \times p} & [0]_{n-p} \end{bmatrix}$ with $A = \begin{bmatrix} 0 & L_0 \\ -L_0^T & [0]_{p-1} \end{bmatrix}$ and $N = \begin{bmatrix} L'_0 \\ [0]_{(p-1) \times (n-p)} \end{bmatrix}$. Therefore, using the block decomposition of matrices of V explained in Remark 3, we find that:

- For every $L \in M_{1,n-1}(\mathbb{K})$, there is a unique³ $A_L \in V$ of the form

$$A_L = \begin{bmatrix} 0 & L & 0 \\ -Z\overline{L}^T & \widetilde{C}_1\overline{L} & 0 \\ f(L) & \varphi(L) & 0 \end{bmatrix},$$

and $f : M_{1,n-1}(\mathbb{K}) \rightarrow \mathbb{K}$ and $\varphi : M_{1,n-1}(\mathbb{K}) \rightarrow M_{1,n-1}(\mathbb{K})$ are linear maps.

²Setting $L_1 := 0$ if $q = 1$, and $C_1 := 0$ if $p = 1$.

³As the map K from the beginning of Section 4.2 is one-to-one.

Let $U \in V_m$, which we write as a block-triangular matrix $U = \begin{bmatrix} [?]_{n-1-i_s} & [?]_{n-1-i_s, i_s} \\ [0]_{i_s, n-1-i_s} & R_s A \end{bmatrix}$ with $A \in A_s(\mathbb{K})$. With the respective structures of V_{ul} and V_m and the fact that V contains no matrix with column space $\text{span}(e_{n+1})$, we know that V contains a unique matrix of the form $\begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ ? & ? & 0 \end{bmatrix}$. Since $Q \times \begin{bmatrix} A & [0]_{(i_s-1) \times 1} \\ [0]_{1 \times (i_s-1)} & 0 \end{bmatrix} = \begin{bmatrix} R_s A & [0]_{(i_s-1) \times 1} \\ L_1 R_s^{-1}(R_s A) & 0 \end{bmatrix}$, the structure of V_{lr} yields that the above matrix of V has $\begin{bmatrix} ? & \widetilde{L}_1 Z^{-1} U & 0 \end{bmatrix}$ as last row. Therefore:

- For every $U \in V_m$, there is a unique $E_U \in V$ of the form

$$E_U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ h(U) & \widetilde{L}_1 Z^{-1} U & 0 \end{bmatrix}.$$

We know that some matrix of V has $[1 \ 0 \ \dots \ 0]^T$ as last column. Summing it with a well-chosen matrix of type A_L , we deduce:

- The subspace V contains a matrix

$$J = \begin{bmatrix} a & 0 & 1 \\ C'_1 & ? & 0 \\ b & L'_1 & 0 \end{bmatrix} \quad \text{with } (a, b) \in \mathbb{K}^2 \text{ and } (L'_1, C'_1) \in M_{1, n-1}(\mathbb{K}) \times M_{n-1, 1}(\mathbb{K}).$$

With the above matrices A_L and J , we find that $\dim(e_1^T V) \geq n$. We already knew that $\dim V = \binom{n+1}{2}$ and $\dim V_{lr} = \binom{n}{2}$, hence the rank theorem shows that the map \mathcal{J} from Section 4.2 yields an isomorphism from the subspace of all matrices of V with zero as first row to V_{lr} . Using the structure of V_{lr} with the same method as in the definition of the A_L matrices, we thus find one last important class of matrices in V :

- For every $C \in M_{n-1, 1}(\mathbb{K})$, there is a unique $B_C \in V$ of the form

$$B_C = \begin{bmatrix} 0 & 0 & 0 \\ \psi(C) & 0 & C \\ g(C) & -\alpha \overline{C}^T (Z^{-1})^T & \widetilde{L}_1 Z^{-1} C \end{bmatrix}$$

and $g : M_{n-1, 1}(\mathbb{K}) \rightarrow \mathbb{K}$ and $\psi : M_{n-1, 1}(\mathbb{K}) \rightarrow M_{n-1, 1}(\mathbb{K})$ are linear maps.

Remark 4. The above matrices span V : a straightforward computation shows indeed that the linear subspaces $\{A_L \mid L \in M_{1,n-1}(\mathbb{K})\}$, $\{B_C \mid C \in M_{n-1,1}(\mathbb{K})\}$, $\{E_U \mid U \in V_m\}$ and $\text{span}(J)$ are independent, and the sum of their dimensions is $(n-1) + (n-1) + \binom{n-1}{2} + 1 = \binom{n+1}{2} = \dim V$.

From now on, our main task is to refine our understanding of the matrices of the types A_L , B_C , E_U and J : the basic strategy is to form well-chosen linear combinations of those special matrices and use the fact that none of them may have a non-zero eigenvalue. Most of the time, we will simply apply the fact that both V and V^T act totally intransitively on \mathbb{K}^{n+1} . Let us start by considering the maps φ and ψ in the A_L and B_C matrices.

Claim 2. *The maps φ and ψ are scalar multiples of the identity.*

Proof. Let $C \in M_{n-1,1}(\mathbb{K})$ and $L \in M_{1,n-1}(\mathbb{K})$. Denote by x (resp. y) the vector of $\text{span}(e_2, \dots, e_n)$ with coordinate matrix C (resp. L^T) in the basis (e_2, \dots, e_n) . We prove that

$$LC = 0 \Rightarrow (\varphi(L)C = 0 \text{ and } L\psi(C) = 0). \quad (5)$$

Assume that $LC = 0$. Notice then that both A_L and B_C stabilize the plane $\text{span}(x, e_{n+1})$ and that the respective matrices of their induced endomorphisms in the basis (x, e_{n+1}) are $\begin{bmatrix} 0 & 0 \\ \varphi(L)C & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ t_1 & t_2 \end{bmatrix}$ for some $(t_1, t_2) \in \mathbb{K}^2$. Since V has a trivial spectrum, we deduce that

$$\forall \lambda \in \mathbb{K}, \quad \begin{vmatrix} 1 & 1 \\ t_1 + \lambda \varphi(L)C & 1 + t_2 \end{vmatrix} \neq 0,$$

hence $\varphi(L)C = 0$.

Similarly, notice that A_L^T and B_C^T both stabilize $\text{span}(e_1, y)$ and the respective matrices of their induced endomorphisms in the basis (e_1, y) are $\begin{bmatrix} 0 & s_1 \\ 1 & s_2 \end{bmatrix}$ and $\begin{bmatrix} 0 & L\psi(C) \\ 0 & 0 \end{bmatrix}$ for some $(s_1, s_2) \in \mathbb{K}^2$. With the above line of reasoning, we deduce that $L\psi(C) = 0$.

We may now conclude. For the non-degenerate bilinear mapping $(L, C) \mapsto LC$ on $M_{1,n-1}(\mathbb{K}) \times M_{n-1,1}(\mathbb{K})$, we deduce from (5) that φ stabilizes the orthogonal subspace of every linear hyperplane of $M_{1,n-1}(\mathbb{K})$, hence φ stabilizes every 1-dimensional linear subspace of $M_{1,n-1}(\mathbb{K})$, which shows that φ is a scalar multiple of the identity. With the same line of reasoning, we see that ψ is also a scalar multiple of the identity. \square

We now have two scalars λ and μ such that:

$$\forall L \in M_{1,n-1}(\mathbb{K}), \quad A_L = \begin{bmatrix} 0 & L & 0 \\ -Z\overline{L}^T & \widetilde{C}_1\overline{L} & 0 \\ f(L) & \lambda L & 0 \end{bmatrix}$$

and

$$\forall C \in M_{n-1,1}(\mathbb{K}), \quad B_C = \begin{bmatrix} 0 & 0 & 0 \\ \mu C & 0 & C \\ g(C) & -\alpha \overline{C}^T (Z^{-1})^T & \widetilde{L}_1 Z^{-1} C \end{bmatrix}.$$

Claim 3. *The map h vanishes everywhere on V_m .*

Proof. Choose $t \in \mathbb{K}$ such that $\mu + t \neq 0$ and $a + t \neq 0$ (this is feasible since $\#\mathbb{K} \geq 3$). Remark then that

$$\left\{ \begin{array}{ll} \forall U \in V_m, & E_U(e_1 + t e_{n+1}) = \begin{bmatrix} 0 \\ [0]_{(n-1) \times 1} \\ h(U) \end{bmatrix} \\ \forall C \in M_{n-1,1}(\mathbb{K}), & B_C(e_1 + t e_{n+1}) = \begin{bmatrix} 0 \\ (\mu + t) C \\ ? \end{bmatrix} \\ & J(e_1 + t e_{n+1}) = \begin{bmatrix} a + t \\ ? \\ ? \end{bmatrix}. \end{array} \right.$$

However $V(e_1 + t e_{n+1})$ is a strict linear subspace of \mathbb{K}^{n+1} . Judging from the vectors $B_C(e_1 + t e_{n+1})$ and the vector $J(e_1 + t e_{n+1})$, we deduce that $V(e_1 + t e_{n+1})$ cannot contain e_{n+1} . This shows that $h(U) = 0$ for every $U \in V_m$. \square

It follows that

$$\forall U \in V_m, \quad E_U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ 0 & \widetilde{L}_1 Z^{-1} U & 0 \end{bmatrix}.$$

From there, we need to study the glued and unglued cases separately.

4.4 The case V_{ul} and V_{lr} are glued

In this section, we assume $p = q = n$. In this case, we simply have $\widetilde{L}_1 = L_1$, $\widetilde{C}_1 = C_1$, $Z = R_1 = R_s$, $V_m = Z A_{n-1}(\mathbb{K})$ and $\forall (L, C) \in M_{1,n-1}(\mathbb{K}) \times M_{n-1,1}(\mathbb{K})$, $\overline{L} = L$ and $\overline{C} = C$. Our aim is to prove that V is equivalent to $A_{n+1}(\mathbb{K})$.

Claim 4. *One has*

$$\forall (L, C) \in M_{1,n-1}(\mathbb{K}) \times M_{n-1,1}(\mathbb{K}), \quad f(L) = -L_1 L^T \quad \text{and} \quad g(C) = \mu L_1 Z^{-1} C.$$

Proof. Let $t \in \mathbb{K} \setminus \{-a\}$. Note that $J(e_1 + te_{n+1})$ has $a+t$ as first entry, whereas

$$\left\{ \begin{array}{l} \forall L \in M_{1,n-1}(\mathbb{K}), \quad A_L(e_1 + te_{n+1}) = \begin{bmatrix} 0 \\ -ZL^T \\ f(L) \end{bmatrix} \\ \forall C \in M_{n-1,1}(\mathbb{K}), \quad B_C(e_1 + te_{n+1}) = \begin{bmatrix} 0 \\ (\mu+t)C \\ g(C) + tL_1 Z^{-1} C \end{bmatrix} \end{array} \right.$$

Judging from $J(e_1 + te_{n+1})$, the vector space $V(e_1 + te_{n+1})$ cannot contain $\text{span}(e_2, \dots, e_{n+1})$. Thus $V(e_1 + te_{n+1}) \cap \text{span}(e_2, \dots, e_{n+1}) = \{A_L(e_1 + te_{n+1}) \mid L \in M_{1,n-1}(\mathbb{K})\}$ (since the first space has a dimension lesser than n and obviously contains the second one). Using the B_C matrices, it follows that

$$\forall C \in M_{n-1,1}(\mathbb{K}), \quad g(C) + tL_1 Z^{-1} C = (\mu+t) f(-C^T(Z^{-1})^T).$$

Since this holds for several values of t , we deduce that

$$\forall C \in M_{n-1,1}(\mathbb{K}), \quad g(C) = -\mu f(C^T(Z^{-1})^T) \quad \text{and} \quad L_1 Z^{-1} C = -f(C^T(Z^{-1})^T),$$

which obviously yields the claimed results. \square

Therefore, for any $(L, C, U) \in M_{1,n-1}(\mathbb{K}) \times M_{n-1,1}(\mathbb{K}) \times Z A_{n-1}(\mathbb{K})$, we have

$$A_L = \begin{bmatrix} 0 & L & 0 \\ -ZL^T & C_1 L & 0 \\ -L_1 L^T & \lambda L & 0 \end{bmatrix} \quad ; \quad B_C = \begin{bmatrix} 0 & 0 & 0 \\ \mu C & 0 & C \\ \mu L_1 Z^{-1} C & -\alpha C^T(Z^{-1})^T & L_1 Z^{-1} C \end{bmatrix}$$

and

$$E_U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ 0 & L_1 Z^{-1} U & 0 \end{bmatrix}.$$

Set now

$$T := \begin{bmatrix} 1 & 0 & 0 \\ C_1 & Z & 0 \\ \lambda & L_1 & \alpha \end{bmatrix} \in \mathrm{GL}_{n+1}(\mathbb{K}) \quad \text{and} \quad T' := \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_{n-1} & 0 \\ -\mu & 0 & 1 \end{bmatrix} \in \mathrm{GL}_{n+1}(\mathbb{K}).$$

A straightforward computation shows that, for every $(L, C, U) \in \mathrm{M}_{1,n-1}(\mathbb{K}) \times \mathrm{M}_{n-1,1}(\mathbb{K}) \times (Z \mathrm{A}_{n-1}(\mathbb{K}))$:

$$T^{-1} A_L T' = \begin{bmatrix} 0 & L & 0 \\ -L^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad T^{-1} B_C T' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & Z^{-1}C \\ 0 & -(Z^{-1}C)^T & 0 \end{bmatrix}$$

and

$$T^{-1} E_U T' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Z^{-1}U & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore $T^{-1}VT'$ contains a linear hyperplane of $\mathrm{A}_{n+1}(\mathbb{K})$. Since V acts totally intransitively on \mathbb{K}^{n+1} , this is also the case of $T^{-1}VT'$, hence Lemma 15 shows that $T^{-1}VT' = \mathrm{A}_{n+1}(\mathbb{K})$. We deduce that V is equivalent to $\mathrm{A}_{n+1}(\mathbb{K})$ and may thus be written as $Y \mathrm{A}_{n+1}(\mathbb{K})$ for some $Y \in \mathrm{GL}_{n+1}(\mathbb{K})$, and Lemma 10 yields that Y is non-isotropic. This completes the case where V_{ul} and V_{lr} are glued.

4.5 The case V_{ul} and V_{lr} are unglued

Here, we assume that $p < n$ or $q < n$. Note that this means that $p = 1$ or $q = 1$ or there are several diagonal blocks $R_1 \mathrm{A}_{i_1}(\mathbb{K}), \dots, R_s \mathrm{A}_{i_s}(\mathbb{K})$ in the block decomposition of V_m discussed earlier. Note in particular that $p + q \leq n + 1$.

Our aim is to prove that V is reducible. Since the matrices A_L, B_C, E_U and J span V , it suffices to find a non-trivial linear subspace of \mathbb{K}^{n+1} which is stabilized by all of them. In that prospect, we start by analyzing f and g .

Claim 5. *One has $f = 0$, and $g(C) = 0$ for every $C \in \mathrm{M}_{n-1,1}(\mathbb{K})$ such that $\overline{C} = 0$.*

Proof. We start by proving that

$$\forall L \in \mathrm{M}_{1,n-1}(\mathbb{K}), \quad \overline{L} = 0 \Rightarrow f(L) = 0 \quad \text{and} \quad \forall C \in \mathrm{M}_{n-1,1}(\mathbb{K}), \quad \overline{C} = 0 \Rightarrow g(C) = 0.$$

We choose $t \in \mathbb{K}$ such that $\mu+t \neq 0$ and $a+t \neq 0$. Then, for every $L \in M_{1,n-1}(\mathbb{K})$ such that $\overline{L} = 0$, one has

$$A_L(e_1 + te_{n+1}) = \begin{bmatrix} 0 \\ [0]_{(n-1) \times 1} \\ f(L) \end{bmatrix},$$

hence $f(L) = 0$ with the same argument as in the proof of Claim 3.

Choose now $x \in \mathbb{K}$ such that $\lambda + x \neq 0$ and $x \neq 0$. Then

$$\begin{cases} \forall L \in M_{1,n-1}(\mathbb{K}), & (xe_1 + e_{n+1})^T A_L = \begin{bmatrix} f(L) & (\lambda + x)L & 0 \end{bmatrix} \\ & (xe_1 + e_{n+1})^T J = \begin{bmatrix} ? & [?]_{1 \times (n-1)} & x \end{bmatrix}. \end{cases}$$

Since $V^T(xe_1 + e_{n+1})$ is a strict linear subspace of \mathbb{K}^{n+1} , those matrices show that e_1 cannot belong to $V^T(xe_1 + e_{n+1})$. However

$$\forall C \in M_{n-1,1}(\mathbb{K}), (xe_1 + e_{n+1})^T B_C = \begin{bmatrix} g(C) & -\alpha \overline{C}^T (Z^{-1})^T & \widetilde{L}_1 Z^{-1} C \end{bmatrix}.$$

Therefore, if $\overline{C} = 0$, then $\widetilde{L}_1 Z^{-1} C = 0$ and hence $g(C) = 0$.

Let $L \in M_{1,n-1}(\mathbb{K})$. The column matrix $C := Z\overline{L}^T$ has null entries starting from the p -th, and since $p + q \leq n + 1$, this yields $\overline{C} = 0$. Therefore $g(C) = 0$ and

$$((\mu + t)A_L + B_C)(e_1 + te_{n+1}) = \begin{bmatrix} 0 \\ [0]_{(n-1) \times 1} \\ (\mu + t)f(L) \end{bmatrix}.$$

The above argument then shows that $f(L) = 0$. □

In particular, we have

$$\forall L \in M_{1,n-1}(\mathbb{K}), A_L = \begin{bmatrix} 0 & L & 0 \\ -Z\overline{L}^T & \widetilde{C}_1 \overline{L} & 0 \\ 0 & \lambda L & 0 \end{bmatrix}.$$

We now distinguish between two cases, whether $p < n$ or $p = n$.

Claim 6. *If $p < n$, then $Ve_1 \subset \text{span}(e_1, \dots, e_p)$.*

Proof. Assume that $p < n$.

Write $C'_1 = \begin{bmatrix} c'_1 \\ \vdots \\ c'_{n-1} \end{bmatrix}$. Let $i \in \llbracket p, n-1 \rrbracket$ (note that such an integer exists).

Let $(x, y, z) \in \mathbb{K}^3$ such that $x + \lambda z \neq 0$. Denote by $C''_i \in M_{n-1,1}(\mathbb{K})$ the column matrix with all entries 0 except the i -th which equals 1. Note that, for every $L \in M_{1,n-1}(\mathbb{K})$, both column matrices \widetilde{C}_1 and $Z\overline{L}^T$ have zero entries starting from the p -th: for \widetilde{C}_1 , this comes from its very definition; for $Z\overline{L}^T$, this is obvious if $p = 1$ because then $\overline{L} = 0$, otherwise this comes from the fact that Z stabilizes $\mathbb{K}^{p-1} \times \{0\} \subset \mathbb{K}^{n-1}$ (as $i_1 = p-1$) and that $\overline{L} \in \mathbb{K}^{p-1} \times \{0\}$. It follows that the $(i+1)$ -th row of every A_L matrix is zero. Setting $\gamma := \widetilde{L}_1 Z^{-1} C''_i$, we therefore have:

$$\left\{ \begin{array}{l} \forall L \in M_{1,n-1}(\mathbb{K}), \quad (xe_1 + ye_{i+1} + ze_{n+1})^T A_L = \begin{bmatrix} 0 & (x + \lambda z)L & 0 \end{bmatrix} \\ (xe_1 + ye_{i+1} + ze_{n+1})^T J = \begin{bmatrix} ax + c'_i y + bz & [?]_{1 \times (n-1)} & x \end{bmatrix} \\ (xe_1 + ye_{i+1} + ze_{n+1})^T B_{C''_i} = \begin{bmatrix} \mu y + g(C''_i)z & [?]_{1 \times (n-1)} & y + \gamma z \end{bmatrix} \end{array} \right.$$

Since $V^T(xe_1 + ye_{i+1} + ze_{n+1}) \neq \mathbb{K}^{n+1}$, we deduce that

$$\begin{vmatrix} ax + c'_i y + bz & x \\ \mu y + g(C''_i)z & y + \gamma z \end{vmatrix} = 0.$$

Notice that, with an arbitrary $(y, z) \in \mathbb{K}^2$ being fixed, the above equation is linear in x and has several solutions, hence

$$(c'_i y + bz)(y + \gamma z) = 0 \quad \text{and} \quad a(y + \gamma z) - (\mu y + g(C''_i)z) = 0.$$

Both equations have a degree lesser than or equal to 2 in both variables. Since $\# \mathbb{K} > 2$, we deduce that

$$c'_i = 0 \quad ; \quad c'_i \gamma + b = 0 \quad ; \quad a = \mu \quad \text{and} \quad a\gamma = g(C''_i).$$

Therefore $a = \mu$, $b = 0$ and $c'_p = \dots = c'_{n-1} = 0$. Since Z is non-singular and stabilizes $\mathbb{K}^{p-1} \times \{0\}$, we may thus find $L \in M_{1,n-1}(\mathbb{K})$ such that $C'_1 = Z\overline{L}^T$.

The first column of $A_L + J$ is $\begin{bmatrix} a \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, therefore $a = 0$ (because $A_L + J$ has no non-zero eigenvalue). It follows that $\mu = 0$ and $g(C''_i) = 0$ for every $i \in \llbracket p, n-1 \rrbracket$.

Since g is linear, $g(C) = 0$ whenever $\overline{C} = 0$ (by Claim 5), and $p - 1 < n - q + 1$, we deduce that $g = 0$.

For any matrix of type A_L , B_C , E_U or J , we have therefore found that its first column has null entries starting from the $(p + 1)$ -th. This yields our claim since these matrices span V . \square

Claim 7. *Assume that $p = n$ (and therefore $q = 1$). Then $\lambda = b = 0$ and $L'_1 = 0$.*

This shows that all the matrices A_L , B_C , E_U and J have zero as last row in the case $p = n$.

Proof. Since $q = 1$, one has $\widetilde{L}_1 = 0$, whilst $\overline{C} = 0$ for every $C \in M_{1,n-1}(\mathbb{K})$. This leads to $f = 0$ and $g = 0$ by Claim 5.

Therefore

$$\forall (L, C) \in M_{1,n-1}(\mathbb{K}) \times M_{n-1,1}(\mathbb{K}), \quad A_L = \begin{bmatrix} 0 & L & 0 \\ -ZL^T & ? & 0 \\ 0 & \lambda L & 0 \end{bmatrix} \quad \text{and} \quad B_C = \begin{bmatrix} 0 & 0 & 0 \\ \mu C & 0 & C \\ 0 & 0 & 0 \end{bmatrix}.$$

Write $L'_1 = [l'_1 \ \cdots \ l'_{n-1}]$. Let $i \in \llbracket 1, n-1 \rrbracket$. Denote by $L''_i \in M_{1,n-1}(\mathbb{K})$ the row matrix with all entries zero except the i -th which equals one.

Let $(x, z) \in \mathbb{K}^2$ such that $\mu x + z \neq 0$. Then

$$\left\{ \begin{array}{l} \forall C \in M_{n-1,1}(\mathbb{K}), \quad B_C(xe_1 + e_{i+1} + ze_{n+1}) = \begin{bmatrix} 0 \\ (\mu x + z)C \\ 0 \end{bmatrix} \\ \\ A_{L''_i}(xe_1 + e_{i+1} + ze_{n+1}) = \begin{bmatrix} 1 \\ [?]_{(n-1) \times 1} \\ \lambda \end{bmatrix} \\ \\ J(xe_1 + e_{i+1} + ze_{n+1}) = \begin{bmatrix} ax + z \\ [?]_{(n-1) \times 1} \\ bx + l'_i \end{bmatrix}. \end{array} \right.$$

We deduce that $\begin{vmatrix} 1 & ax + z \\ \lambda & bx + l'_i \end{vmatrix} = 0$. Since, for a given $x \in \mathbb{K}$, this holds for several values of z , we successively deduce that $\lambda = 0$ and $\forall x \in \mathbb{K}, bx + l'_i = 0$, which yields $\lambda = b = l'_i = 0$. Therefore $L'_1 = 0$. \square

In two special cases, we may now conclude that V is reducible: if $p = 1$ then Claim 6 shows that $\text{span}(e_1)$ is stabilized by V ; if $p = n$, then Claims 5 and 7 show that $\text{span}(e_1, \dots, e_n)$ is stabilized by V (indeed, in that case $q = 1$ and hence $\widetilde{L}_1 = 0$ and $\overline{C} = 0$ for every $C \in M_{n-1,1}(\mathbb{K})$).

Assume finally that $1 < p < n$. Then $Ve_1 \subset \text{span}(e_1, \dots, e_p)$ by Claim 6. Note that the change of basis matrix $R = \begin{bmatrix} I_{n+1-q} & 0 \\ 0 & T_1^T \end{bmatrix}$ from Section 4.2 leaves $\text{span}(e_1, \dots, e_p)$ invariant as $p \leq n+1-q$. Therefore we also have $(R^{-1}VR)e_1 \subset \text{span}(e_1, \dots, e_p)$, and some of our recent findings may be summed up as follows:

Proposition 19. *Let V be a maximal subspace of $M_{n+1}(\mathbb{K})$ with a trivial spectrum such that:*

- (i) $Ve_{n+1} = \text{span}(e_1, \dots, e_n)$;
- (ii) *There are lower-triangular non-isotropic matrices $P \in \text{GL}_p(\mathbb{K})$, $P_2 \in \text{GL}_{n_2}(\mathbb{K}), \dots, P_r \in \text{GL}_{n_r}(\mathbb{K})$, with $1 < p < n$, such that $V_{ul} = P A_p(\mathbb{K}) \vee P_2 A_{n_2}(\mathbb{K}) \vee \dots \vee P_r A_{n_r}(\mathbb{K})$.*

Then $Ve_1 \subset \text{span}(e_1, \dots, e_p)$.

Note that the fact that V contains no matrix with column space $\text{span}(e_{n+1})$, our starting point in Section 4.2, is a consequence of assumptions (i) and (ii) of Proposition 19 (using the rank theorem to compute the dimension of V from that of V_{ul} , as in the beginning of Section 4.2).

Now, all we need to complete the unglued case is to show that any V satisfying the assumptions of Proposition 19 is reducible. Let V be such a subspace, with the above notations. Let $x \in \text{span}(e_1, \dots, e_p) \setminus \{0\}$. Recall that the bilinear form $b : (X, Y) \in (\mathbb{K}^p)^2 \mapsto X^T P Y$ is non-isotropic, and hence non-degenerate. Denote by X_0 the matrix of coordinates of x in (e_1, \dots, e_p) . In the hyperplane $H := \{Y \in \mathbb{K}^p : X_0^T Y = 0\}$, we may therefore find a “right-sided orthogonal basis” (f_2, \dots, f_p) , i.e. $b(f_i, f_j) = 0$ for every $(i, j) \in \llbracket 2, p \rrbracket^2$ with $i < j$. We then choose a non-zero vector f_1 such that $b(f_1, f_j) = 0$ for every $j \in \llbracket 2, p \rrbracket$. It follows that (f_1, \dots, f_p) is a basis of \mathbb{K}^p . Denoting by S the matrix of coordinates of (f_1, f_2, \dots, f_p) in (e_1, \dots, e_p) , the matrix $P' := S^T P S$ is lower-triangular and

$$S^T (P A_p(\mathbb{K})) (S^T)^{-1} = P' A_p(\mathbb{K}).$$

Set then $T_2 := \begin{bmatrix} S^T & 0 \\ 0 & I_{n+1-p} \end{bmatrix} \in \mathrm{GL}_{n+1}(\mathbb{K})$ and

$$V' := T_2 V T_2^{-1}.$$

Notice finally that T_2 stabilizes $\mathrm{span}(e_1, \dots, e_n)$, fixes e_{n+1} , and obviously

$$V'_{ul} = P' A_p(\mathbb{K}) \vee P_2 A_{n_2}(\mathbb{K}) \vee \dots \vee P_r A_{n_r}(\mathbb{K}).$$

Thus Proposition 19 applied to V' shows that $V'e_1 \subset \mathrm{span}(e_1, \dots, e_p)$. However S maps $\mathrm{span}(e_2, \dots, e_p)$ to $\mathrm{span}(f_2, \dots, f_p)$, hence $S^T X_0 \in \mathrm{span}(e_1) \setminus \{0\}$. This yields

$$Vx \subset \mathrm{span}(e_1, \dots, e_p).$$

We conclude that $\mathrm{span}(e_1, \dots, e_p)$ is a non-trivial invariant subspace for V , hence V is reducible. This completes our proof of Theorem 3.

5 On large spaces of nilpotent matrices

In this short section, we show that the following famous theorem of Gerstenhaber on linear subspaces of nilpotent matrices is an easy consequence of Theorem 4:

Theorem 20 (Gerstenhaber's theorem). *Let \mathbb{K} be a field with at least three elements, and V be a linear subspace of $M_n(\mathbb{K})$ such that $\dim V = \binom{n}{2}$ and every matrix of V is nilpotent. Then V is similar to $\mathrm{NT}_n(\mathbb{K})$.*

See [6] for the original proof under the more restrictive assumption $\#\mathbb{K} \geq n$, [7] for a very elegant proof using trace maps and a theorem of Jacobson, and [14] for a proof with no restriction on the cardinality of \mathbb{K} .

Proof. The assumptions show that V is a maximal linear subspace of $M_n(\mathbb{K})$ with a trivial spectrum. Then $V \simeq P_1 A_{n_1}(\mathbb{K}) \vee \dots \vee P_p A_{n_p}(\mathbb{K})$ for non-isotropic matrices P_1, \dots, P_p . Since every matrix of V is nilpotent, every matrix of $P_k A_{n_k}(\mathbb{K})$ is nilpotent for every $k \in \llbracket 1, p \rrbracket$.

Let $q \geq 2$ be a positive integer and $P \in \mathrm{GL}_q(\mathbb{K})$, and assume that P is non-isotropic and every element of $P A_q(\mathbb{K})$ is nilpotent. Note that q is odd since $A_q(\mathbb{K})$ contains non-singular matrices when q is even. Then $\mathrm{tr}(PA) = 0$ for every $A \in A_q(\mathbb{K})$, which shows that P is symmetric. Since q is odd and P is non-singular, P is not alternate hence it is congruent to a non-singular diagonal matrix D (even if \mathbb{K} has characteristic 2, see [9, Chapter 35]). Thus $D A_q(\mathbb{K})$

is similar to $PA_q(\mathbb{K})$ and must therefore have a trivial spectrum. Finally, set $K := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $A := \begin{bmatrix} K & [0]_{2 \times (q-2)} \\ [0]_{(q-2) \times 2} & [0]_{q-2} \end{bmatrix} \in A_q(\mathbb{K})$, and note that DA is obviously non-nilpotent, a contradiction.

Returning to V , we deduce that $n_1 = \dots = n_p = 1$ hence $V \simeq \text{NT}_n(\mathbb{K})$. \square

6 On the exceptional case of \mathbb{F}_2

In the proof of Theorem 4, we have repeatedly used the assumption that the field \mathbb{K} had at least 3 elements. The reader will therefore not be surprised by the following counterexample which shows that Theorem 4 fails for the field \mathbb{F}_2 . Remark first that there is no non-isotropic matrix in $\text{GL}_3(\mathbb{F}_2)$ (since every 3-dimensional quadratic form over a finite field is isotropic), hence no maximal linear subspace of $M_3(\mathbb{F}_2)$ with a trivial spectrum has the form $PA_3(\mathbb{F}_2)$.

Consider the following matrices of $M_3(\mathbb{F}_2)$:

$$A := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad ; \quad B := \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Using the identities $\forall x \in \mathbb{F}_2, x + x = 0$ and $x^2 = x$, a straightforward computation yields

$$\forall (x, y, z) \in \mathbb{F}_2^3, \quad \det(I_3 + xA + yB + zC) = 1.$$

Therefore the 3-dimensional subspace $V := \text{span}(A, B, C)$ has a trivial spectrum. The fact that $A + B$ is non-singular shows however that V is irreducible. If V were reducible indeed, then there would exist a 1-dimensional subspace W of $M_2(\mathbb{F}_2)$ such that $V \simeq \{0\} \vee W$ or $V \simeq W \vee \{0\}$, and in both cases every matrix of V would be singular.

The classification of the irreducible maximal subspaces of $M_n(\mathbb{F}_2)$ with a trivial spectrum thus remains an unresolved issue.

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